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CHAPTER 1

MULTILINEAR ALGEBRA

1.1 Background

We will list below some definitions and theorems that are part of the curriculum of a standard theory-based sophomore level course in linear algebra. (Such a course is a prerequisite for reading these notes.) A *vector space* is a set, V, the elements of which we will refer to as *vectors*. It is equipped with two vector space operations:
Vector space addition. Given two vectors, v_1 and v_2, one can add them to get a third vector, $v_1 + v_2$.
Scalar multiplication. Given a vector, v, and a real number, λ, one can multiply v by λ to get a vector, λv.

These operations satisfy a number of standard rules: associativity, commutativity, distributive laws, etc. which we assume you're familiar with. (See exercise 1 below.) In addition we'll assume you're familiar with the following definitions and theorems.

1. *The zero vector.* This vector has the property that for every vector, v, $v + 0 = 0 + v = v$ and $\lambda v = 0$ if λ is the real number, zero.

2. *Linear independence.* A collection of vectors, v_i, $i = 1, \ldots, k$, is *linearly* independent if the map

$$\mathbb{R}^k \to V \,, \quad (c_1, \ldots, c_k) \to c_1 v_1 + \cdots + c_k v_k \tag{1.1.1}$$

is $1 - 1$.

3. The *spanning property.* A collection of vectors, v_i, $i = 1, \ldots, k$, *spans* V if the map (1.1.1) is onto.

4. The notion of *basis.* The vectors, v_i, in items 2 and 3 are a basis of V if they span V and are linearly independent; in other words, if the map (1.1.1) is bijective. This means that every vector, v, can be written uniquely as a sum

$$v = \sum c_i v_i \,. \tag{1.1.2}$$

5. The *dimension* of a vector space. If V possesses a basis, v_i, $i = 1, \dots, k$, V is said to be *finite dimensional*, and k is, by definition, the *dimension* of V. (It is a theorem that this definition is legitimate: every basis has to have the same number of vectors.) In this chapter all the vector spaces we'll encounter will be finite dimensional.

6. A subset, U, of V is a *subspace* if it's vector space in its own right, i.e., for v, v_1 and v_2 in U and λ in $\mathbb{R}$, λv and $v_1 + v_2$ are in U.

7. Let V and W be vector spaces. A map, $A : V \to W$ is *linear* if, for v, v_1 and v_2 in V and $\lambda \in \mathbb{R}$

$$A(\lambda v) = \lambda A v \tag{1.1.3}$$

and

$$A(v_1 + v_2) = Av_1 + Av_2 \,. \tag{1.1.4}$$

8. The *kernel of A.* This is the set of vectors, v, in V which get mapped by A into the zero vector in W. By (1.1.3) and (1.1.4) this set is a subspace of V. We'll denote it by "Ker A".

9. *The image of A.* By (1.1.3) and (1.1.4) the image of A, which we'll denote by "Im A", is a subspace of W. The following is an important rule for keeping track of the dimensions of Ker A and Im A.

$$\dim V = \dim \operatorname{Ker} A + \dim \operatorname{Im} A \,. \tag{1.1.5}$$

Example 1. The map (1.1.1) is a linear map. The v_i's span V if its image is V and the v_i's are linearly independent if its kernel is just the zero vector in $\mathbb{R}^k$.

10. *Linear mappings and matrices.* Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ a basis of W. Then by (1.1.2) Av_j can be written uniquely as a sum,

$$Av_j = \sum_{i=1}^{m} c_{i,j} w_i \,, \quad c_{i,j} \in \mathbb{R} \,. \tag{1.1.6}$$

The $m \times n$ matrix of real numbers, $[c_{i,j}]$, is the *matrix* associated with A. Conversely, given such an $m \times n$ matrix, there is a unique linear map, A, with the property (1.1.6).

11. An *inner product* on a vector space is a map

$$B : V \times V \to \mathbb{R}$$

having the three properties below.

(a) For vectors, v, v_1, v_2 and w and $\lambda \in \mathbb{R}$

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

and

$$B(\lambda v, w) = \lambda B(v, w)\,.$$

(b) For vectors, v and w,

$$B(v, w) = B(w, v)\,.$$

(c) For every vector, v

$$B(v, v) \geq 0\,.$$

Moreover, if $v \neq 0$, $B(v, v)$ is positive.

Notice that by property (b), property (a) is equivalent to

$$B(w, \lambda v) = \lambda B(w, v)$$

and

$$B(w, v_1 + v_2) = B(w, v_1) + B(w, v_2)\,.$$

The items on the list above are just a few of the topics in linear algebra that we're assuming our readers are familiar with. We've highlighted them because they're easy to state. However, understanding them requires a heavy dollop of that indefinable quality "mathematical sophistication", a quality which will be in heavy demand in the next few sections of this chapter. We will also assume that our readers are familiar with a number of more low-brow linear algebra notions: matrix multiplication, row and column operations on matrices, transposes of matrices, determinants of $n \times n$ matrices, inverses of matrices, Cramer's rule, recipes for solving systems of linear equations, etc. (See §1.1 and 1.2 of Munkres' book for a quick review of this material.)

Exercises.

1. Our basic example of a vector space in this course is $\mathbb{R}^n$ equipped with the vector addition operation

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$

and the scalar multiplication operation

$$\lambda(a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n)\,.$$

Check that these operations satisfy the axioms below.

(a) Commutativity: $v + w = w + v$.

(b) Associativity: $u + (v + w) = (u + v) + w$.

(c) For the zero vector, $0 = (0, \ldots, 0)$, $v + 0 = 0 + v$.

(d) $v + (-1)v = 0$.

(e) $1v = v$.

(f) Associative law for scalar multiplication: $(ab)v = a(bv)$.

(g) Distributive law for scalar addition: $(a + b)v = av + bv$.

(h) Distributive law for vector addition: $a(v + w) = av + aw$.

2. Check that the standard basis vectors of $\mathbb{R}^n$: $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc. *are* a basis.

3. Check that the standard inner product on $\mathbb{R}^n$

$$B((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum_{i=1}^{n} a_i b_i$$

is an inner product.

1.2 Quotient spaces and dual spaces

In this section we will discuss a couple of items which are frequently, but not always, covered in linear algebra courses, but which we'll need for our treatment of multilinear algebra in §§1.1.3 – 1.1.8.

The quotient spaces of a vector space

Let V be a vector space and W a vector subspace of V. A *W-coset* is a set of the form

$$v + W = \{v + w\,,\ w \in W\}\,.$$

It is easy to check that if $v_1 - v_2 \in W$, the cosets, $v_1 + W$ and $v_2 + W$, coincide while if $v_1 - v_2 \notin W$, they are disjoint. Thus the W-cosets decompose V into a *disjoint* collection of subsets of V. We will denote this collection of sets by V/W.

One defines a vector addition operation on V/W by defining the sum of two cosets, $v_1 + W$ and $v_2 + W$ to be the coset

$$v_1 + v_2 + W \tag{1.2.1}$$

and one defines a scalar multiplication operation by defining the scalar multiple of $v + W$ by λ to be the coset

$$\lambda v + W\,. \tag{1.2.2}$$

It is easy to see that these operations are well defined. For instance, suppose $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. Then $v_1 - v_1'$ and $v_2 - v_2'$ are in W; so $(v_1 + v_2) - (v_1' + v_2')$ is in W and hence $v_1 + v_2 + W = v_1' + v_2' + W$.

These operations make V/W into a vector space, and one calls this space the *quotient space* of V by W.

We define a mapping

$$\pi : V \to V/W \tag{1.2.3}$$

by setting $\pi(v) = v + W$. It's clear from (1.2.1) and (1.2.2) that π is a linear mapping, and that it maps V to V/W. Moreover, for every coset, $v + W$, $\pi(v) = v + W$; so the mapping, π, is onto. Also note that the zero vector in the vector space, V/W, is the zero coset, $0 + W = W$. Hence v is in the kernel of π if $v + W = W$, i.e., $v \in W$. In other words the kernel of π is W.

In the definition above, V and W don't have to be finite dimensional, but if they are, then

$$\dim V/W = \dim V - \dim W\,. \tag{1.2.4}$$

by (1.1.5).

The following, which is easy to prove, we'll leave as an exercise.

Proposition 1.2.1. *Let U be a vector space and $A : V \to U$ a linear map. If $W \subset \operatorname{Ker} A$ there exists a unique linear map, $A^\# : V/W \to U$ with property, $A = A^\# \circ \pi$.*

The dual space of a vector space

We'll denote by V^* the set of all linear functions, $\ell : V \to \mathbb{R}$. If ℓ_1 and ℓ_2 are linear functions, their sum, $\ell_1 + \ell_2$, is linear, and if ℓ is a linear function and λ is a real number, the function, $\lambda\ell$, is linear. Hence V^* is a vector space. One calls this space the *dual space* of V.

Suppose V is n-dimensional, and let $e_1, \dots, e_n$ be a basis of V. Then every vector, $v \in V$, can be written uniquely as a sum

$$v = c_1 e_1 + \cdots + c_n e_n \quad c_i \in \mathbb{R}\,.$$

Let

$$e_i^*(v) = c_i\,. \tag{1.2.5}$$

If $v = c_1 e_1 + \cdots + c_n e_n$ and $v' = c_1' e_1 + \cdots + c_n' e_n$ then $v + v' = (c_1 + c_1')e_1 + \cdots + (c_n + c_n')e_n$, so

$$e_i^*(v + v') = c_i + c_i' = e_i^*(v) + e_i^*(v')\,.$$

This shows that $e_i^*(v)$ is a linear function of v and hence $e_i^* \in V^*$.

Claim: e_i^*, $i = 1, \dots, n$ is a basis of V^*.

Proof. First of all note that by (1.2.5)

$$e_i^*(e_j) = \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}\,. \tag{1.2.6}$$

If $\ell \in V^*$ let $\lambda_i = \ell(e_i)$ and let $\ell' = \sum \lambda_i e_i^*$. Then by (1.2.6)

$$\ell'(e_j) = \sum \lambda_i e_i^*(e_j) = \lambda_j = \ell(e_j)\,, \tag{1.2.7}$$

i.e., ℓ and ℓ' take identical values on the basis vectors, e_j. Hence $\ell = \ell'$.

Suppose next that $\sum \lambda_i e_i^* = 0$. Then by (1.2.6), $\lambda_j = (\sum \lambda_i e_i^*)(e_j) = 0$ for all $j = 1, \dots, n$. Hence the e_j^*'s are linearly independent.

□

Let V and W be vector spaces and $A : V \to W$, a linear map. Given $\ell \in W^*$ the composition, $\ell \circ A$, of A with the linear map, $\ell : W \to \mathbb{R}$, is linear, and hence is an element of V^*. We will denote this element by $A^*\ell$, and we will denote by

$$A^* : W^* \to V^*$$

the map, $\ell \to A^*\ell$. It's clear from the definition that

$$A^*(\ell_1 + \ell_2) = A^*\ell_1 + A^*\ell_2$$

and that

$$A^*\lambda\ell = \lambda A^*\ell \,,$$

i.e., that A^* is linear.

Definition. A^* is the transpose of the mapping A.

We will conclude this section by giving a matrix description of A^*. Let $e_1, \dots, e_n$ be a basis of V and $f_1, \dots, f_m$ a basis of W; let $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ be the dual bases of V^* and W^*. Suppose A is defined in terms of $e_1, \dots, e_n$ and $f_1, \dots, f_m$ by the $m \times n$ matrix, $[a_{i,j}]$, i.e., suppose

$$Ae_j = \sum a_{i,j} f_i \,.$$

Claim. A^* is defined, in terms of $f_1^*, \dots, f_m^*$ and $e_1^*, \dots, e_n^*$ by the transpose matrix, $[a_{j,i}]$.

Proof. Let

$$A^* f_i^* = \sum c_{j,i} e_j^* \,.$$

Then

$$A^* f_i^*(e_j) = \sum_k c_{k,i} e_k^*(e_j) = c_{j,i}$$

by (1.2.6). On the other hand

$$A^* f_i^*(e_j) \quad = \quad f_i^*(Ae_j) = f_i^* \left(\sum a_{k,j} f_k \right) = \sum_k a_{k,j} f_i^*(f_k) = a_{i,j}$$

so $a_{i,j} = c_{j,i}$.

□

Exercises.

1. Let V be an n-dimensional vector space and W a k-dimensional subspace. Show that there exists a basis, $e_1, \dots, e_n$ of V with the property that $e_1, \dots, e_k$ is a basis of W. *Hint:* Induction on $n-k$. To start the induction suppose that $n-k=1$. Let $e_1, \dots, e_{n-1}$ be a basis of W and e_n any vector in $V-W$.

2. In exercise 1 show that the vectors $f_i = \pi(e_{k+i})$, $i = 1, \dots, n-k$ are a basis of V/W.

3. In exercise 1 let U be the linear span of the vectors, e_{k+i}, $i = 1, \dots, n-k$.

Show that the map

$$U \to V/W\,, \quad u \to \pi(u)\,,$$

is a vector space isomorphism, i.e., show that it maps U bijectively onto V/W.

4. Let U, V and W be vector spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^* A^*$.

5. Let $V = \mathbb{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0)\,;\, x_1 \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

(a) Show that the W-cosets are the lines, $x_2 = a$, parallel to the x_1-axis.

(b) Show that the sum of the cosets, "$x_2 = a$" and "$x_2 = b$" is the coset "$x_2 = a + b$".

(c) Show that the scalar multiple of the coset, "$x_2 = c$" by the number, λ, is the coset, "$x_2 = \lambda c$".

6. (a) Let $(V^*)^*$ be the dual of the vector space, V^*. For every $v \in V$, let $\mu_v : V^* \to \mathbb{R}$ be the function, $\mu_v(\ell) = \ell(v)$. Show that the μ_v is a linear function on V^*, i.e., an element of $(V^*)^*$, and show that the map

$$\mu : V \to (V^*)^* \quad v \to \mu_v \tag{1.2.8}$$

is a linear map of V into $(V^*)^*$.

(b) Show that the map (1.2.8) is bijective. (*Hint:* $\dim(V^*)^* = \dim V^* = \dim V$, so by (1.1.5) it suffices to show that (1.2.8) is injective.) Conclude that there is a *natural* identification of V with $(V^*)^*$, i.e., that V and $(V^*)^*$ are two descriptions of the same object.

7. Let W be a vector subspace of V and let

$$W^\perp = \{\ell \in V^*\,,\ \ell(w) = 0 \text{ if } w \in W\}\,.$$

Show that $W^\perp$ is a subspace of V^* and that its dimension is equal to $\dim V - \dim W$. (*Hint:* By exercise 1 we can choose a basis, $e_1, \ldots, e_n$ of V such that $e_1, \ldots e_k$ is a basis of W. Show that $e^*_{k+1}, \ldots, e^*_n$ is a basis of $W^\perp$.) $W^\perp$ is called the *annihilator* of W in V^*.

8. Let V and V' be vector spaces and $A : V \to V'$ a linear map. Show that if W is the kernel of A there exists a linear map, $B : V/W \to V'$, with the property: $A = B \circ \pi$, π being the map (1.2.3). In addition show that this linear map is injective.

9. Let W be a subspace of a finite-dimensional vector space, V. From the inclusion map, $\iota : W^\perp \to V^*$, one gets a transpose map,

$$\iota^* : (V^*)^* \to (W^\perp)^*$$

and, by composing this with (1.2.8), a map

$$\iota^* \circ \mu : V \to (W^\perp)^*\,.$$

Show that this map is onto and that its kernel is W. Conclude from exercise 8 that there is a *natural* bijective linear map

$$\nu : V/W \to (W^\perp)^*$$

with the property $\nu \circ \pi = \iota^* \circ \mu$. In other words, V/W and $(W^\perp)^*$ are two descriptions of the same object. (This shows that the "quotient space" operation and the "dual space" operation are closely related.)

10. Let V_1 and V_2 be vector spaces and $A : V_1 \to V_2$ a linear map. Verify that for the transpose map: $A^* : V_2^* \to V_1^*$

$$\operatorname{Ker} A^* = (\operatorname{Im} A)^\perp$$

and

$$\operatorname{Im} A^* = (\operatorname{Ker} A)^\perp\,.$$

11. (a) Let $B : V \times V \to \mathbb{R}$ be an inner product on V. For $v \in V$ let

$$\ell_v : V \to \mathbb{R}$$

be the function: $\ell_v(w) = B(v, w)$. Show that ℓ_v is linear and show that the map

$$L : V \to V^*\,, \quad v \to \ell_v \tag{1.2.9}$$

is a linear mapping.

(b) Prove that this mapping is bijective. (*Hint:* Since $\dim V = \dim V^*$ it suffices by (1.1.5) to show that its kernel is zero. Now note that if $v \neq 0$ $\ell_v(v) = B(v, v)$ is a positive number.) Conclude that if V has an inner product one gets from it a *natural* identification of V with V^*.

12. Let V be an n-dimensional vector space and $B : V \times V \to \mathbb{R}$ an inner product on V. A basis, $e_1, \ldots, e_n$ of V is *orthonormal* is

$$B(e_i, e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.2.10}$$

(a) Show that an orthonormal basis exists. *Hint:* By induction let e_i, $i = 1, \ldots, k$ be vectors with the property (1.2.10) and let v be a vector which is not a linear combination of these vectors. Show that the vector

$$w = v - \sum B(e_i, v)e_i$$

is non-zero and is orthogonal to the e_i's. Now let $e_{k+1} = \lambda w$, where $\lambda = B(w, w)^{-\frac{1}{2}}$.

(b) Let $e_1, \ldots e_n$ and $e'_1, \ldots e'_n$ be two orthogonal bases of V and let

$$e'_j = \sum a_{i,j} e_i\,. \tag{1.2.11}$$

Show that

$$\sum a_{i,j} a_{i,k} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \tag{1.2.12}$$

(c) Let A be the matrix $[a_{i,j}]$. Show that (1.2.12) can be written more compactly as the matrix identity

$$AA^t = I \tag{1.2.13}$$

where I is the identity matrix.

(d) Let $e_1, \dots, e_n$ be an orthonormal basis of V and $e_1^*, \dots, e_n^*$ the dual basis of V^*. Show that the mapping (1.2.9) is the mapping, $Le_i = e_i^*$, $i = 1, \dots n$.

1.3 Tensors

Let V be an n-dimensional vector space and let V^k be the set of all k-tuples, $(v_1,\dots,v_k)$, $v_i \in V$. A function

$$T : V^k \to \mathbb{R}$$

is said to be linear in its i^{th} variable if, when we fix vectors, $v_1,\dots,v_{i-1}$, $v_{i+1},\dots,v_k$, the map

$$v \in V \to T(v_1,\dots,v_{i-1},v,v_{i+1},\dots,v_k) \tag{1.3.1}$$

is linear in V. If T is linear in its i^{th} variable for $i=1,\dots,k$ it is said to be *k-linear*, or alternatively is said to be a *k-tensor*. We denote the set of all k-tensors by $\mathcal{L}^k(V)$. We will agree that 0-tensors are just the real numbers, that is $\mathcal{L}^0(V)=\mathbb{R}$.

Let T_1 and T_2 be functions on V^k. It is clear from (1.3.1) that if T_1 and T_2 are k-linear, so is T_1+T_2. Similarly if T is k-linear and λ is a real number, λT is k-linear. Hence $\mathcal{L}^k(V)$ is a vector space. Note that for $k=1$, "k-linear" just means "linear", so $\mathcal{L}^1(V)=V^*$.

Let $I=(i_1,\dots i_k)$ be a sequence of integers with $1 \le i_r \le n$, $r=1,\dots,k$. We will call such a sequence *a multi-index* of length k. For instance the multi-indices of length 2 are the square arrays of pairs of integers

$$(i,j)\,,\ 1 \le i,j \le n$$

and there are exactly n^2 of them.

Exercise.

Show that there are exactly n^k multi-indices of length k.

Now fix a basis, $e_1,\dots,e_n$, of V and for $T \in \mathcal{L}^k(V)$ let

$$T_I = T(e_{i_1},\dots,e_{i_k}) \tag{1.3.2}$$

for every multi-index I of length k.

Proposition 1.3.1. *The T_I's determine T, i.e., if T and T' are k-tensors and $T_I=T'_I$ for all I, then $T=T'$.*

Proof. By induction on n. For $n = 1$ we proved this result in § 1.1. Let's prove that if this assertion is true for $n-1$, it's true for n. For each e_i let T_i be the $(k-1)$-tensor

$$(v_1, \dots, v_{n-1}) \to T(v_1, \dots, v_{n-1}, e_i)\,.$$

Then for $v = c_1 e_1 + \cdots c_n e_n$

$$T(v_1, \dots, v_{n-1}, v) = \sum c_i T_i(v_1, \dots, v_{n-1})\,,$$

so the T_i's determine T. Now apply induction.

□

The tensor product operation

If T_1 is a k-tensor and T_2 is an ℓ-tensor, one can define a $k+\ell$-tensor, $T_1 \otimes T_2$, by setting

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell})\,.$$

This tensor is called *the tensor product* of T_1 and T_2. We note that if T_1 or T_2 is a 0-tensor, i.e., scalar, then tensor product with *it* is just scalar multiplication by *it*, that is $a \otimes T = T \otimes a = aT$ $(a \in \mathbb{R}\,,\ T \in \mathcal{L}^k(V))$.

Similarly, given a k-tensor, T_1, an ℓ-tensor, T_2 and an m-tensor, T_3, one can define a $(k+\ell+m)$-tensor, $T_1 \otimes T_2 \otimes T_3$ by setting

(1.3.3) $$T_1 \otimes T_2 \otimes T_3(v_1, \dots, v_{k+\ell+m}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell}) T_3(v_{k+\ell+1}, \dots, v_{k+\ell+m})\,.$$

Alternatively, one can define (1.3.3) by defining it to be the tensor product of $T_1 \otimes T_2$ and T_3 or the tensor product of T_1 and $T_2 \otimes T_3$. It's easy to see that both these tensor products are identical with (1.3.3):

(1.3.4) $$(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3) = T_1 \otimes T_2 \otimes T_3\,.$$

We leave for you to check that if λ is a real number

(1.3.5) $$\lambda(T_1 \otimes T_2) = (\lambda T_1) \otimes T_2 = T_1 \otimes (\lambda T_2)$$

and that the left and right distributive laws are valid: For $k_1 = k_2$,

(1.3.6) $$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

and for $k_2 = k_3$

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3 \,. \tag{1.3.7}$$

A particularly interesting tensor product is the following. For $i = 1, \dots, k$ let $\ell_i \in V^*$ and let

$$T = \ell_1 \otimes \dots \otimes \ell_k \,. \tag{1.3.8}$$

Thus, by definition,

$$T(v_1, \dots, v_k) = \ell_1(v_1) \dots \ell_k(v_k) \,. \tag{1.3.9}$$

A tensor of the form (1.3.9) is called a *decomposable* k-tensor. These tensors, as we will see, play an important role in what follows. In particular, let $e_1, \dots, e_n$ be a basis of V and $e_1^*, \dots, e_n^*$ the dual basis of V^*. For every multi-index, I, of length k let

$$e_I^* = e_{i_1}^* \otimes \dots \otimes e_{i_k}^* \,.$$

Then if J is another multi-index of length k,

$$e_I^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1\,, & I = J \\ 0\,, & I \neq J \end{cases} \tag{1.3.10}$$

by (1.2.6), (1.3.8) and (1.3.9). From (1.3.10) it's easy to conclude

Theorem 1.3.2. *The e_I^*'s are a basis of $\mathcal{L}^k(V)$.*

Proof. Given $T \in \mathcal{L}^k(V)$, let

$$T' = \sum T_I e_I^*$$

where the T_I's are defined by (1.3.2). Then

$$T'(e_{j_1}, \dots, e_{j_k}) = \sum T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J \tag{1.3.11}$$

by (1.3.10); however, by Proposition 1.3.1 the T_J's determine T, so $T' = T$. This proves that the e_I^*'s are a spanning set of vectors for $\mathcal{L}^k(V)$. To prove they're a basis, suppose

$$\sum C_I e_I^* = 0$$

for constants, $C_I \in \mathbb{R}$. Then by (1.3.11) with $T' = 0$, $C_J = 0$, so the e_I^*'s are linearly independent.

□

As we noted above there are exactly n^k multi-indices of length k and hence n^k basis vectors in the set, $\{e_I^*\}$, so we've proved

Corollary. $\dim \mathcal{L}^k(V) = n^k$.

The pull-back operation

Let V and W be finite dimensional vector spaces and let $A : V \to W$ be a linear mapping. If $T \in \mathcal{L}^k(W)$, we define

$$A^*T : V^k \to \mathbb{R}$$

to be the function

$$A^*T(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)\,. \tag{1.3.12}$$

It's clear from the linearity of A that this function is linear in its i^{th} variable for all i, and hence is k-tensor. We will call A^*T the *pull-back* of T by the map, A.

Proposition 1.3.3. *The map*

$$A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)\,, \quad T \to A^*T\,, \tag{1.3.13}$$

is a linear mapping.

We leave this as an exercise. We also leave as an exercise the identity

$$A^*(T_1 \otimes T_2) = A^*T_1 \otimes A^*T_2 \tag{1.3.14}$$

for $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$. Also, if U is a vector space and $B : U \to V$ a linear mapping, we leave for you to check that

$$(AB)^*T = B^*(A^*T) \tag{1.3.15}$$

for all $T \in \mathcal{L}^k(W)$.

Exercises.

1. Verify that there are exactly n^k multi-indices of length k.

2. Prove Proposition 1.3.3.

3. Verify (1.3.14).

4. Verify (1.3.15).

5. Let $A : V \to W$ be a linear map. Show that if ℓ_i, $i = 1, \ldots, k$ are elements of W^*

$$A^*(\ell_1 \otimes \cdots \otimes \ell_k) = A^*\ell_1 \otimes \cdots \otimes A^*\ell_k\,.$$

Conclude that A^* maps decomposable k-tensors to decomposable k-tensors.

6. Let V be an n-dimensional vector space and ℓ_i, $i = 1, 2$, elements of V^*. Show that $\ell_1 \otimes \ell_2 = \ell_2 \otimes \ell_1$ if and only if ℓ_1 and ℓ_2 are linearly dependent. (*Hint:* Show that if ℓ_1 and ℓ_2 are linearly independent there exist vectors, v_i, $i =, 1, 2$ in V with property

$$\ell_i(v_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} .$$

Now compare $(\ell_1 \otimes \ell_2)(v_1, v_2)$ and $(\ell_2 \otimes \ell_1)(v_1, v_2)$.) Conclude that if $\dim V \geq 2$ the tensor product operation isn't commutative, i.e., it's usually not true that $\ell_1 \otimes \ell_2 = \ell_2 \otimes \ell_1$.

7. Let T be a k-tensor and v a vector. Define $T_v : V^{k-1} \to \mathbb{R}$ to be the map

$$T_v(v_1, \ldots, v_{k-1}) = T(v, v_1, \ldots, v_{k-1})\,. \tag{1.3.16}$$

Show that T_v is a $(k-1)$-tensor.

8. Show that if T_1 is an r-tensor and T_2 is an s-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2\,.$$

9. Let $A : V \to W$ be a linear map mapping $v \in V$ to $w \in W$. Show that for $T \in \mathcal{L}^k(W)$, $A^*(T_w) = (A^*T)_v$.

1.4 Alternating k-tensors

We will discuss in this section a class of k-tensors which play an important role in multivariable calculus. In this discussion we will need some standard facts about the "permutation group". For those of you who are already familiar with this object (and I suspect most of you are) you can regard the paragraph below as a chance to re-familiarize yourselves with these facts.

Permutations

Let $\sum_k$ be the k-element set: $\{1, 2, \dots, k\}$. A *permutation of order k* is a bijective map, $\sigma : \sum_k \to \sum_k$. Given two permutations, σ_1 and σ_2, their *product*, $\sigma_1\sigma_2$, is the composition of σ_1 and σ_2, i.e., the map,

$$i \to \sigma_1(\sigma_2(i)),$$

and for every permutation, σ, one denotes by σ^{-1} the inverse permutation:

$$\sigma(i) = j \Leftrightarrow \sigma^{-1}(j) = i.$$

Let S_k be the set of all permutations of order k. One calls S_k the *permutation group* of $\sum_k$ or, alternatively, the *symmetric group on k letters.*

Check:

There are $k!$ elements in S_k.

For every $1 \le i < j \le k$, let $\tau = \tau_{i,j}$ be the permutation

$$\begin{aligned} \tau(i) &= j \\ \tau(j) &= i \\ \tau(\ell) &= \ell, \quad \ell \neq i, j. \end{aligned} \tag{1.4.1}$$

τ is called a *transposition*, and if $j = i + 1$, τ is called an *elementary transposition.*

Theorem 1.4.1. *Every permutation can be written as a product of finite number of transpositions.*

Proof. Induction on k: "$k = 2$" is obvious. *The induction step: "$k-1$" implies "k":* Given $\sigma \in S_k$, $\sigma(k) = i \Leftrightarrow \tau_{ik}\sigma(k) = k$. Thus $\tau_{ik}\sigma$ is, in effect, a permutation of $\sum_{k-1}$. By induction, $\tau_{ik}\sigma$ can be written as a product of transpositions, so

$$\sigma = \tau_{ik}(\tau_{ik}\sigma)$$

can be written as a product of transpositions.

□

Theorem 1.4.2. *Every transposition can be written as a product of elementary transpositions.*

Proof. Let $\tau = \tau_{ij}$, $i < j$. With i fixed, argue by induction on j. Note that for $j > i + 1$

$$\tau_{ij} = \tau_{j-1,j}\tau_{i,j-1}\tau_{j-1,j} .$$

Now apply induction to $\tau_{i,j-1}$.

□

Corollary. *Every permutation can be written as a product of elementary transpositions.*

The sign of a permutation

Let $x_1, \ldots, x_k$ be the coordinate functions on $\mathbb{R}^k$. For $\sigma \in S_k$ we define

$$(1.4.2) \qquad (-1)^\sigma = \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j} .$$

Notice that the numerator and denominator in this expression are identical up to sign. Indeed, if $p = \sigma(i) < \sigma(j) = q$, the term, $x_p - x_q$ occurs once and just once in the numerator and one and just one in the denominator; and if $q = \sigma(i) > \sigma(j) = p$, the term, $x_p - x_q$, occurs once and just once in the numerator and its negative, $x_q - x_p$, once and just once in the numerator. Thus

$$(1.4.3) \qquad (-1)^\sigma = \pm 1 .$$

Claim:

For $\sigma, \tau \in S_k$

$$(-1)^{\sigma\tau} = (-1)^{\sigma}(-1)^{\tau} . \tag{1.4.4}$$

Proof. By definition,

$$(-1)^{\sigma\tau} = \prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_i - x_j} .$$

We write the right hand side as a product of

$$\prod_{i<j} \frac{x_{\tau(i)} - x_{\tau(j)}}{x_i - x_j} = (-1)^{\tau} \tag{1.4.5}$$

and

$$\prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_{\tau(i)} - x_{\tau(j)}} \tag{1.4.6}$$

For $i < j$, let $p = \tau(i)$ and $q = \tau(j)$ when $\tau(i) < \tau(j)$ and let $p = \tau(j)$ and $q = \tau(i)$ when $\tau(j) < \tau(i)$. Then

$$\frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_{\tau(i)} - x_{\tau(j)}} = \frac{x_{\sigma(p)} - x_{\sigma(q)}}{x_p - x_q}$$

(i.e., if $\tau(i) < \tau(j)$, the numerator and denominator on the right *equal* the numerator and denominator on the left and, if $\tau(j) < \tau(i)$ are *negatives* of the numerator and denominator on the left). Thus (1.4.6) becomes

$$\prod_{p<q} \frac{x_{\sigma(p)} - x_{\sigma(q)}}{x_p - x_q} = (-1)^{\sigma} .$$

□

We'll leave for you to check that if τ is a transposition, $(-1)^{\tau} = -1$ and to conclude from this:

Proposition 1.4.3. *If σ is the product of an odd number of transpositions, $(-1)^{\sigma} = -1$ and if σ is the product of an even number of transpositions $(-1)^{\sigma} = +1$.*

Alternation

Let V be an n-dimensional vector space and $T \in \mathcal{L}^*(v)$ a k-tensor. If $\sigma \in S_k$, let $T^\sigma \in \mathcal{L}^*(V)$ be the k-tensor

$$T^\sigma(v_1, \ldots, v_k) = T(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(k)}) . \tag{1.4.7}$$

Proposition 1.4.4. *1. If $T = \ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in V^*$, then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.*

2. The map, $T \in \mathcal{L}^k(V) \to T^\sigma \in \mathcal{L}^k(V)$ is a linear map.

3. $T^{\sigma\tau} = (T^\tau)^\sigma$.

Proof. To prove 1, we note that by (1.4.7)

$$\begin{aligned} &(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma(v_1, \ldots, v_k) \\ &= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}) . \end{aligned}$$

Setting $\sigma^{-1}(i) = q$, the i^{th} term in this product is $\ell_{\sigma(q)}(v_q)$; so the product can be rewritten as

$$\ell_{\sigma(1)}(v_1) \ldots \ell_{\sigma(k)}(v_k)$$

or

$$(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)})(v_1, \ldots, v_k) .$$

The proof of 2 we'll leave as an exercise.

Proof of 3: By item 2, it suffices to check 3 for decomposable tensors. However, by 1

$$\begin{aligned} (\ell_1 \otimes \cdots \otimes \ell_k)^{\sigma\tau} &= \ell_{\sigma\tau(1)} \otimes \cdots \otimes \ell_{\sigma\tau(k)} \\ &= (\ell_{\tau(1)} \otimes \cdots \otimes \ell_{\tau(k)})^\sigma \\ &= ((\ell_1 \otimes \cdots \otimes \ell)^\tau)^\sigma . \end{aligned}$$

Definition 1.4.5. *$T \in \mathcal{L}^k(V)$ is* alternating *if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.*

We will denote by $\mathcal{A}^k(V)$ the set of all alternating k-tensors in $\mathcal{L}^k(V)$. By item 2 of Proposition 1.4.4 this set is a vector subspace of $\mathcal{L}^k(V)$.

It is not easy to write down simple examples of alternating k-tensors; however, there is a method, called the *alternation operation*, for constructing such tensors: Given $T \in \mathcal{L}^*(V)$ let

$$\operatorname{Alt} T = \sum_{\tau \in S_k} (-1)^\tau T^\tau \,. \tag{1.4.8}$$

We claim

Proposition 1.4.6. *For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$,*

1. $(\operatorname{Alt} T)^\sigma = (-1)^\sigma \operatorname{Alt} T$
2. *if* $T \in \mathcal{A}^k(V)$, $\operatorname{Alt} T = k!T$.
3. $\operatorname{Alt} T^\sigma = (\operatorname{Alt} T)^\sigma$
4. *the map*

$$\operatorname{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V) \,,\; T \to \operatorname{Alt}(T)$$

is linear.

Proof. To prove 1 we note that by Proposition (1.4.4):

$$\begin{aligned} (\operatorname{Alt} T)^\sigma &= \sum (-1)^\tau (T^{\sigma\tau}) \\ &= (-1)^\sigma \sum (-1)^{\sigma\tau} T^{\sigma\tau} \,. \end{aligned}$$

But as τ runs over S_k, $\sigma\tau$ runs over S_k, and hence the right hand side is $(-1)^\sigma \operatorname{Alt}(T)$.

□

Proof of 2. If $T \in \mathcal{A}^k$

$$\begin{aligned} \operatorname{Alt} T &= \sum (-1)^\tau T^\tau \\ &= \sum (-1)^\tau (-1)^\tau T \\ &= k!\,T \,. \end{aligned}$$

□

Proof of 3.

$$\begin{aligned} \operatorname{Alt} T^\sigma &= \sum (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \operatorname{Alt} T = (\operatorname{Alt} T)^\sigma \,. \end{aligned}$$

□

Finally, item 4 is an easy corollary of item 2 of Proposition 1.4.4. □

We will use this alternation operation to construct a basis for $\mathcal{A}^k(V)$. First, however, we require some notation:

Let $I = (i_1, \ldots, i_k)$ be a multi-index of length k.

Definition 1.4.7. *1.* *I is* repeating *if $i_r = i_s$ for some $r \neq s$.*

2. *I is strictly increasing if $i_1 < i_2 < \cdots < i_r$.*

3. *For $\sigma \in S_k$, $I^\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$.*

Remark: If I is non-repeating there is a unique $\sigma \in S_k$ so that I^σ is strictly increasing.

Let $e_1, \ldots, e_n$ be a basis of V and let

$$e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$$

and

$$\psi_I = \mathrm{Alt}\,(e_I^*)\,.$$

Proposition 1.4.8. *1.* $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

2. *If I is repeating, $\psi_I = 0$.*

3. *If I and J are strictly increasing,*

$$\psi_I(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases} .$$

Proof. To prove 1 we note that $(e_I^*)^\sigma = e_{I^\sigma}^*$; so

$$\mathrm{Alt}\,(e_{I^\sigma}^*) = \mathrm{Alt}\,(e_I^*)^\sigma = (-1)^\sigma \mathrm{Alt}\,(e_I^*)\,.$$

□

Proof of 2: Suppose $I = (i_1, \ldots, i_k)$ with $i_r = i_s$ for $r \neq s$. Then if $\tau = \tau_{i_r, i_s}$, $e_I^* = e_{I^r}^*$ so

$$\psi_I = \psi_{I^r} = (-1)^\tau \psi_I = -\psi_I\,.$$

□

Proof of 3: By definition

$$\psi_I(e_{j_1}, \ldots, e_{j_k}) = \sum (-1)^\tau e^*_{I^\tau}(e_{j_1}, \ldots, e_{j_k}) \,.$$

But by (1.3.10)

$$e^*_{I^\tau}(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1 \text{ if } I^\tau = J \\ 0 \text{ if } I^\tau \neq J \end{cases} . \tag{1.4.9}$$

Thus if I and J are strictly increasing, I^τ is strictly increasing if and only if $I^\tau = I$, and (1.4.9) is non-zero if and only if $I = J$.

□

Now let T be in $\mathcal{A}^k$. By Proposition 1.3.2,

$$T = \sum a_J e^*_J \,, \quad a_J \in \mathbb{R} \,.$$

Since

$$\begin{aligned} k!T &= \operatorname{Alt}(T) \\ T &= \frac{1}{k!} \sum a_J \operatorname{Alt}(e^*_J) = \sum b_J \psi_J \,. \end{aligned}$$

We can discard all repeating terms in this sum since they are zero; and for every non-repeating term, J, we can write $J = I^\sigma$, where I is strictly increasing, and hence $\psi_J = (-1)^\sigma \psi_I$.

Conclusion:

We can write T as a sum

$$T = \sum c_I \psi_I \,, \tag{1.4.10}$$

with I's strictly increasing.

Claim.

The c_I's are unique.

Proof. For J strictly increasing

$$T(e_{j_1}, \dots, e_{j_k}) = \sum c_I \psi_I(e_{j_1}, \dots, e_{j_k}) = c_J \,. \tag{1.4.11}$$

By (1.4.10) the ψ_I's, I strictly increasing, are a spanning set of vectors for $\mathcal{A}^k(V)$, and by (1.4.11) they are linearly independent, so we've proved

Proposition 1.4.9. *The alternating tensors, ψ_I, I strictly increasing, are a basis for $\mathcal{A}^k(V)$.*

Thus $\dim \mathcal{A}^k(V)$ is equal to the number of strictly increasing multi-indices, I, of length k. We leave for you as an exercise to show that this number is equal to

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \text{“ } n \text{ choose } k\text{”} \tag{1.4.12}$$

if $1 \leq k \leq n$.

□

Hint: Show that every strictly increasing multi-index of length k determines a k element subset of $\{1, \dots, n\}$ and vice-versa.

Note also that if $k > n$ every multi-index

$$I = (i_1, \dots, i_k)$$

of length k has to be repeating: $i_r = i_s$ for some $r \neq s$ since the i_p's lie on the interval $1 \leq i \leq n$. Thus by Proposition 1.4.6

$$\psi_I = 0$$

for all multi-indices of length $k > 0$ and

$$\mathcal{A}^k = \{0\} \,. \tag{1.4.13}$$

Exercises.

1. Show that there are exactly $k!$ permutations of order k. *Hint:* Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$, $1 \leq i \leq k$. Show that $\tau_{ik}\sigma$ leaves k fixed and hence is, in effect, a permutation of $\sum_{k-1}$.

2. Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$ and deduce from this Proposition 1.4.3.

3. Prove assertion 2 in Proposition 1.4.4.

4. Prove that $\dim \mathcal{A}^k(V)$ is given by (1.4.12).

5. Verify that for $i < j-1$

$$\tau_{i,j} = \tau_{j-1,j}\tau_{i,j-1}, \tau_{j-1,j} \,.$$

6. For $k = 3$ show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

7. Let $\sigma \in S_k$ be the "cyclic" permutation

$$\sigma(i) = i + 1 \,, \quad i = 1, \dots, k-1$$

and $\sigma(k) = 1$. Show explicitly how to write σ as a product of transpositions and compute $(-1)^\sigma$. *Hint:* Same hint as in exercise 1.

8. In exercise 7 of Section 3 show that if T is in $\mathcal{A}^k$, T_v is in $\mathcal{A}^{k-1}$. Show in addition that for $v, w \in V$ and $T \in \mathcal{A}^k$, $(T_v)_w = -(T_w)_v$.

9. Let $A : V \to W$ be a linear mapping. Show that if T is in $\mathcal{A}^k(W)$, A^*T is in $\mathcal{A}^k(V)$.

10. In exercise 9 show that if T is in $\mathcal{L}^k(W)$, $\mathrm{Alt}\,(A^*T) = A^*(\mathrm{Alt}\,(T))$, i.e., show that the "Alt" operation commutes with the pull-back operation.

1.5 The space, $\Lambda^k(V^*)$

In § 1.4 we showed that the image of the alternation operation, $\text{Alt}: \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is $\mathcal{A}^k(V)$. In this section we will compute the kernel of Alt.

Definition 1.5.1. *A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in V^*$, is redundant if for some index, i, $\ell_i = \ell_{i+1}$.*

Let $\mathcal{I}^k$ be the linear span of the set of reductant k-tensors.

Note that for $k = 1$ the notion of redundant doesn't really make sense; a single vector $\ell \in \mathcal{L}^1(V^*)$ can't be "redundant" so we decree

$$\mathcal{I}^1(V) = \{0\}\,.$$

Proposition 1.5.2. *If $T \in \mathcal{I}^k$, $\text{Alt}\,(T) = 0$.*

Proof. Let $T = \ell_k \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\tau = \tau_{i,i+1}$, $T^\tau = T$ and $(-1)^\tau = -1$. Hence $\text{Alt}\,(T) = \text{Alt}\,(T^\tau) = \text{Alt}\,(T)^\tau = -\text{Alt}\,(T)$; so $\text{Alt}\,(T) = 0$. □

To simplify notation let's abbreviate $\mathcal{L}^k(V)$, $\mathcal{A}^k(V)$ and $\mathcal{I}^k(V)$ to $\mathcal{L}^k$, $\mathcal{A}^k$ and $\mathcal{I}^k$.

Proposition 1.5.3. *If $T \in \mathcal{I}^r$ and $T' \in \mathcal{L}^s$ then $T \otimes T'$ and $T' \otimes T$ are in $\mathcal{I}^{r+s}$.*

Proof. We can assume that T and T' are decomposable, i.e., $T = \ell_1 \otimes \cdots \otimes \ell_r$ and $T' = \ell_1' \otimes \cdots \otimes \ell_s'$ and that T is redundant: $\ell_i = \ell_{i+1}$. Then

$$T \otimes T' = \ell_1 \otimes \cdots \ell_{i-1} \otimes \ell_i \otimes \ell_i \otimes \cdots \ell_r \otimes \ell_1' \otimes \cdots \otimes \ell_s'$$

is redundant and hence in $\mathcal{I}^{r+s}$. The argument for $T' \otimes T$ is similar. □

Proposition 1.5.4. *If $T \in \mathcal{L}^k$ and $\sigma \in S_k$, then*

$$T^\sigma = (-1)^\sigma T + S \tag{1.5.1}$$

where S is in $\mathcal{I}^k$.

Proof. We can assume T is decomposable, i.e., $T = \ell_1 \otimes \cdots \otimes \ell_k$. Let's first look at the simplest possible case: $k = 2$ and $\sigma = \tau_{1,2}$. Then

$$\begin{aligned} T^\sigma - (-)^\sigma T &= \ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1 \\ &= ((\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2)/2\,, \end{aligned}$$

and the terms on the right are redundant, and hence in $\mathcal{I}^2$. Next let k be arbitrary and $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-2}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$. Then

$$T - (-1)^\sigma T = T_1 \otimes (\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \otimes T_2$$

is in $\mathcal{I}^k$ by Proposition 1.5.3 and the computation above.
The general case: By Theorem 1.4.2, σ can be written as a product of m elementary transpositions, and we'll prove (1.5.1) by induction on m.

We've just dealt with the case $m = 1$.
The induction step: "$m-1$" implies "m". Let $\sigma = \tau\beta$ where β is a product of $m-1$ elementary transpositions and τ is an elementary transposition. Then

$$\begin{aligned} T^\sigma = (T^\beta)^\tau &= (-1)^\tau T^\beta + \cdots \\ &= (-1)^\tau (-1)^\beta T + \cdots \\ &= (-1)^\sigma T + \cdots \end{aligned}$$

where the "dots" are elements of $\mathcal{I}^k$, and the induction hypothesis was used in line 2.

□

Corollary. *If $T \in \mathcal{L}^k$, the*

(1.5.2) $$\mathrm{Alt}\,(T) = k!T + W\,,$$

where W is in $\mathcal{I}^k$.

Proof. By definition $\mathrm{Alt}\,(T) = \sum (-1)^\sigma T^\sigma$, and by Proposition 1.5.4, $T^\sigma = (-1)^\sigma T + W_\sigma$, with $W_\sigma \in \mathcal{I}^k$. Thus

$$\begin{aligned} \mathrm{Alt}\,(T) &= \sum (-1)^\sigma (-1)^\sigma T + \sum (-1)^\sigma W_\sigma \\ &= k!T + W \end{aligned}$$

where $W = \sum (-1)^\sigma W_\sigma$.

□

Corollary. *$\mathcal{I}^k$ is the kernel of* Alt .

Proof. We've already proved that if $T \in \mathcal{I}^k$, $\mathrm{Alt}\,(T) = 0$. To prove the converse assertion we note that if $\mathrm{Alt}\,(T) = 0$, then by (1.5.2)

$$T = -\tfrac{1}{k!} W\,.$$

with $W \in \mathcal{I}^k$. □

Putting these results together we conclude:

Theorem 1.5.5. *Every element, T, of $\mathcal{L}^k$ can be written uniquely as a sum, $T = T_1 + T_2$ where $T_1 \in \mathcal{A}^k$ and $T_2 \in \mathcal{I}^k$.*

Proof. By (1.5.2), $T = T_1 + T_2$ with

$$T_1 = \tfrac{1}{k!}\mathrm{Alt}\,(T)$$

and

$$T_2 = -\tfrac{1}{k!} W\,.$$

To prove that this decomposition is unique, suppose $T_1 + T_2 = 0$, with $T_1 \in \mathcal{A}^k$ and $T_2 \in \mathcal{I}^k$. Then

$$0 = \mathrm{Alt}\,(T_1 + T_2) = k! T_1$$

so $T_1 = 0$, and hence $T_2 = 0$.

□

Let

(1.5.3) $$\Lambda^k(V^*) = \mathcal{L}^k(V^*)/\mathcal{I}^k(V^*)\,,$$

i.e., let $\Lambda^k = \Lambda^k(V^*)$ be the quotient of the vector space $\mathcal{L}^k$ by the subspace, $\mathcal{I}^k$, of $\mathcal{L}^k$. By (1.2.3) one has a linear map:

(1.5.4) $$\pi : \mathcal{L}^k \to \Lambda^k\,, \quad T \to T + \mathcal{I}^k$$

which is onto and has $\mathcal{I}^k$ as kernel. We claim:

Theorem 1.5.6. *The map, π, maps $\mathcal{A}^k$ bijectively onto Λ^k.*

Proof. By Theorem 1.5.5 every $\mathcal{I}^k$ coset, $T + \mathcal{I}^k$, contains a unique element, T_1, of $\mathcal{A}^k$. Hence for every element of Λ^k there is a unique element of $\mathcal{A}^k$ which gets mapped onto it by π.

□

Remark. Since Λ^k and $\mathcal{A}^k$ are isomorphic as vector spaces many treatments of multilinear algebra avoid mentioning Λ^k, reasoning that $\mathcal{A}^k$ is a perfectly good substitute for it and that one should, if possible, not make two different definitions for what is essentially the same object. This is a justifiable point of view (and is the point of view taken by Spivak and Munkres[1]). There are, however, some advantages to distinguishing between A^k and Λ^k, as we'll see in § 1.6.

Exercises.

1. A k-tensor, T, $\in \mathcal{L}^k(V)$ is *symmetric* if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set, $\mathcal{S}^k(V)$, of symmetric k tensors is a vector subspace of $\mathcal{L}^k(V)$.

2. Let $e_1, \dots, e_n$ be a basis of V. Show that every symmetric 2-tensor is of the form

$$\sum a_{ij} e_i^* \otimes e_j^*$$

where $a_{i,j} = a_{j,i}$ and $e_1^*, \dots, e_n^*$ are the dual basis vectors of V^*.

3. Show that if T is a symmetric k-tensor, then for $k \geq 2$, T is in $\mathcal{I}^k$. *Hint:* Let σ be a transposition and deduce from the identity, $T^\sigma = T$, that T has to be in the kernel of Alt.

4. *Warning:* In general $\mathcal{S}^k(V) \neq \mathcal{I}^k(V)$. Show, however, that if $k = 2$ these two spaces are equal.

5. Show that if $\ell \in V^*$ and $T \in \mathcal{I}^{k-2}$, then $\ell \otimes T \otimes \ell$ is in $\mathcal{I}^k$.

6. Show that if ℓ_1 and ℓ_2 are in V^* and T is in $\mathcal{I}^{k-2}$, then $\ell_1 \otimes T \otimes \ell_2 + \ell_2 \otimes T \otimes \ell_1$ is in $\mathcal{I}^k$.

7. Given a permutation $\sigma \in S_k$ and $T \in \mathcal{I}^k$, show that $T^\sigma \in \mathcal{I}^k$.

8. Let $\mathcal{W}$ be a subspace of $\mathcal{L}^k$ having the following two properties.

(a) For $S \in \mathcal{S}^2(V)$ and $T \in \mathcal{L}^{k-2}$, $S \otimes T$ is in $\mathcal{W}$.

(b) For T in $\mathcal{W}$ and $\sigma \in S_k$, T^σ is in $\mathcal{W}$.

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[1] and by the author of these notes in his book with Alan Pollack, "Differential Topology"

Show that $\mathcal{W}$ has to contain $\mathcal{I}^k$ and conclude that $\mathcal{I}^k$ is the smallest subspace of $\mathcal{L}^k$ having properties a and b.

9. Show that there is a bijective linear map

$$\alpha : \Lambda^k \to \mathcal{A}^k$$

with the property

$$\alpha\pi(T) = \frac{1}{k!}\mathrm{Alt}\,(T) \tag{1.5.5}$$

for all $T \in \mathcal{L}^k$, and show that α is the inverse of the map of $\mathcal{A}^k$ onto Λ^k described in Theorem 1.5.6 (*Hint:* §1.2, exercise 8).

10. Let V be an n-dimensional vector space. Compute the dimension of $S^k(V)$. *Some hints:*

(a) Introduce the following symmetrization operation on tensors $T \in \mathcal{L}^k(V)$:

$$\mathrm{Sym}(T) = \sum_{\tau \in S_k} T^\tau \,.$$

Prove that this operation has properties 2, 3 and 4 of Proposition 1.4.6 and, as a substitute for property 1, has the property: $(\mathrm{Sym}T)^\sigma = \mathrm{Sym}T$.

(b) Let $\varphi_I = \mathrm{Sym}(e_I^*)$, $e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_n}^*$. Prove that $\{\varphi_I,\ I$ non-decreasing$\}$ form a basis of $S^k(V)$.

(c) Conclude from (b) that $\dim S^k(V)$ is equal to the number of non-decreasing multi-indices of length k: $1 \le i_1 \le i_2 \le \cdots \le \ell_k \le n$.

(d) Compute this number by noticing that

$$(i_1, \ldots, i_n) \to (i_1 + 0, i_2 + 1, \ldots, i_k + k - 1)$$

is a bijection between the set of these non-decreasing multi-indices and the set of increasing multi-indices $1 \le j_1 < \cdots < j_k \le n + k - 1$.

1.6 The wedge product

The tensor algebra operations on the spaces, $\mathcal{L}^k(V)$, which we discussed in Sections 1.2 and 1.3, i.e., the "tensor product operation" and the "pull-back" operation, give rise to similar operations on the spaces, Λ^k. We will discuss in this section the analogue of the tensor product operation. As in § 4 we'll abbreviate $\mathcal{L}^k(V)$ to $\mathcal{L}^k$ and $\Lambda^k(V)$ to Λ^k when it's clear which "V" is intended.

Given $\omega_i \in \Lambda^{k_i}$, $i = 1, 2$ we can, by (1.5.4), find a $T_i \in \mathcal{L}^{k_i}$ with $\omega_i = \pi(T_i)$. Then $T_1 \otimes T_2 \in \mathcal{L}^{k_1+k_2}$. Let

$$\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2) \in \Lambda^{k_1+k_2}\,. \tag{1.6.1}$$

Claim.

This wedge product is well defined, i.e., doesn't depend on our choices of T_1 and T_2.

Proof. Let $\pi(T_1) = \pi(T_1') = \omega_1$. Then $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}$, so

$$T_1' \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2\,.$$

But $W_1 \in \mathcal{I}^{k_1}$ implies $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}$ and this implies:

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2)\,.$$

A similar argument shows that (1.6.1) is well-defined independent of the choice of T_2.

□

More generally let $\omega_i \in \Lambda^{k_i}$, $i = 1, 2, 3$, and let $\omega_i = \pi(T_i)$, $T_i \in \mathcal{L}^{k_i}$. Define

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \in \Lambda^{k_1+k_2+k_3}$$

by setting

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \pi(T_1 \otimes T_2 \otimes T_3)\,.$$

As above it's easy to see that this is well-defined independent of the choice of T_1, T_2 and T_3. It is also easy to see that this triple wedge product is just the wedge product of $\omega_1 \wedge \omega_2$ with ω_3 or, alternatively, the wedge product of ω_1 with $\omega_2 \wedge \omega_3$, i.e.,

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3). \tag{1.6.2}$$

We leave for you to check:

For $\lambda \in \mathbb{R}$

$$\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2) \tag{1.6.3}$$

and verify the two distributive laws:

$$(\omega_1 + \omega_2) \wedge \omega_3 \quad = \quad \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \tag{1.6.4}$$

and

$$\omega_1 \wedge (\omega_2 + \omega_3) \quad = \quad \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3\,. \tag{1.6.5}$$

As we noted in § 1.4, $\mathcal{I}^k = \{0\}$ for $k = 1$, i.e., there are no non-zero "redundant" k tensors in degree $k = 1$. Thus

$$\Lambda^1(V^*) = V^* = \mathcal{L}^1(V^*). \tag{1.6.6}$$

A particularly interesting example of a wedge product is the following. Let $\ell_i \in V^* = \Lambda^1(V^*)$, $i = 1, \ldots, k$. Then if $T = \ell_1 \otimes \cdots \otimes \ell_k$

$$\ell_1 \wedge \cdots \wedge \ell_k = \pi(T) \in \Lambda^k(V^*)\,. \tag{1.6.7}$$

We will call (1.6.7) a *decomposable element* of $\Lambda^k(V^*)$.

We will prove that these elements satisfy the following wedge product identity. For $\sigma \in S_k$:

$$\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k\,. \tag{1.6.8}$$

Proof. For every $T \in \mathcal{L}^k$, $T = (-1)^\sigma T + W$ for some $W \in I^k$ by Proposition 1.5.4. Therefore since $\pi(W) = 0$

$$\pi(T^\sigma) = (-1)^\sigma \pi(T)\,. \tag{1.6.9}$$

In particular, if $T = \ell_1 \otimes \cdots \otimes \ell_k$, $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$, so

$$\begin{aligned} \pi(T^\sigma) &= \ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \pi(T) \\ &= (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k\,. \end{aligned}$$

□

In particular, for ℓ_1 and $\ell_2 \in V^*$

$$\ell_1 \wedge \ell_2 = -\ell_2 \wedge \ell_1 \tag{1.6.10}$$

and for ℓ_1, ℓ_2 and $\ell_3 \in V^*$

$$\ell_1 \wedge \ell_2 \wedge \ell_3 = -\ell_2 \wedge \ell_1 \wedge \ell_3 = \ell_2 \wedge \ell_3 \wedge \ell_1 \,. \tag{1.6.11}$$

More generally, it's easy to deduce from (1.6.8) the following result (which we'll leave as an exercise).

Theorem 1.6.1. *If $\omega_1 \in \Lambda^r$ and $\omega_2 \in \Lambda^s$ then*

$$\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1 \,. \tag{1.6.12}$$

Hint: It suffices to prove this for decomposable elements i.e., for $\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r$ and $\omega_2 = \ell_1' \wedge \cdots \wedge \ell_s'$. Now make rs applications of (1.6.10).

Let $e_1, \ldots, e_n$ be a basis of V and let $e_1^*, \ldots, e_n^*$ be the dual basis of V^*. For every multi-index, I, of length k,

$$e_{i_1}^* \wedge \cdots e_{i_k}^* = \pi(e_I^*) = \pi(e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*) \,. \tag{1.6.13}$$

Theorem 1.6.2. *The elements (1.6.13), with I strictly increasing, are basis vectors of Λ^k.*

Proof. The elements

$$\psi_I = \mathrm{Alt}\,(e_I^*) \,,\ I \text{ strictly increasing,}$$

are basis vectors of $\mathcal{A}^k$ by Proposition 3.6; so their images, $\pi(\psi_I)$, are a basis of Λ^k. But

$$\begin{aligned} \pi(\psi_I) &= \pi \sum (-1)^\sigma (e_I^*)^\sigma \\ &= \sum (-1)^\sigma \pi(e_I^*)^\sigma \\ &= \sum (-1)^\sigma (-1)^\sigma \pi(e_I^*) \\ &= k!\pi(e_I^*) \,. \end{aligned}$$

□

Exercises:

1. Prove the assertions (1.6.3), (1.6.4) and (1.6.5).

2. Verify the multiplication law, (1.6.12) for wedge product.

3. Given $\omega \in \Lambda^r$ let ω^k be the k-fold wedge product of ω with itself, i.e., let $\omega^2 = \omega \wedge \omega$, $\omega^3 = \omega \wedge \omega \wedge \omega$, etc.

(a) Show that if r is odd then for $k > 1$, $\omega^k = 0$.

(b) Show that if ω is decomposable, then for $k > 1$, $\omega^k = 0$.

4. If ω and μ are in Λ^{2r} prove:

$$(\omega + \mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell} \,.$$

Hint: As in freshman calculus prove this binomial theorem by induction using the identity: $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.

5. Let ω be an element of Λ^2. By definition the *rank* of ω is k if $\omega^k \neq 0$ and $\omega^{k+1} = 0$. Show that if

$$\omega = e_1 \wedge f_1 + \cdots + e_k \wedge f_k$$

with $e_i, f_i \in V^*$, then ω is of rank $\leq k$. *Hint:* Show that

$$\omega^k = k! e_1 \wedge f_1 \wedge \cdots \wedge e_k \wedge f_k \,.$$

6. Given $e_i \in V^*$, $i = 1, \ldots, k$ show that $e_1 \wedge \cdots \wedge e_k \neq 0$ if and only if the e_i's are linearly independent. *Hint:* Induction on k.

1.7 The interior product

We'll describe in this section another basic product operation on the spaces, $\Lambda^k(V^*)$. As above we'll begin by defining this operator on the $\mathcal{L}^k(V)$'s. Given $T \in \mathcal{L}^k(V)$ and $\mathrm{v} \in V$ let $\iota_{\mathrm{v}} T$ be the be the $(k-1)$-tensor which takes the value

(1.7.1)

$$\iota_{\mathrm{v}} T(\mathrm{v}_1, \dots, \mathrm{v}_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(\mathrm{v}_1, \dots, \mathrm{v}_{r-1}, \mathrm{v}, \mathrm{v}_r, \dots, \mathrm{v}_{k-1})$$

on the $k-1$-tuple of vectors, $\mathrm{v}_1, \dots, \mathrm{v}_{k-1}$, i.e., in the r^{th} summand on the right, v gets inserted between v_{r-1} and v_r. (In particular the first summand is $T(\mathrm{v}, \mathrm{v}_1, \dots, \mathrm{v}_{k-1})$ and the last summand is $(-1)^{k-1} T(\mathrm{v}_1, \dots, \mathrm{v}_{k-1}, \mathrm{v})$.) It's clear from the definition that if $\mathrm{v} = \mathrm{v}_1 + \mathrm{v}_2$

$$\iota_{\mathrm{v}} T = \iota_{\mathrm{v}_1} T + \iota_{\mathrm{v}_2} T \,, \tag{1.7.2}$$

and if $T = T_1 + T_2$

$$\iota_{\mathrm{v}} T = \iota_{\mathrm{v}} T_1 + \iota_{\mathrm{v}} T_2 \,, \tag{1.7.3}$$

and we will leave for you to verify by inspection the following two lemmas:

Lemma 1.7.1. *If T is the decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$ then*

$$\iota_{\mathrm{v}} T = \sum (-1)^{r-1} \ell_r(\mathrm{v}) \ell_1 \otimes \cdots \otimes \widehat{\ell}_r \otimes \cdots \otimes \ell_k \tag{1.7.4}$$

where the "cap" over ℓ_r means that it's deleted from the tensor product ,

and

Lemma 1.7.2. *If $T_1 \in \mathcal{L}^p$ and $T_2 \in \mathcal{L}^q$*

$$\iota_{\mathrm{v}}(T_1 \otimes T_2) = \iota_{\mathrm{v}} T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_{\mathrm{v}} T_2 \,. \tag{1.7.5}$$

We will next prove the important identity

$$\iota_{\mathrm{v}}(\iota_{\mathrm{v}} T) = 0 \,. \tag{1.7.6}$$

Proof. It suffices by linearity to prove this for decomposable tensors and since (1.7.6) is trivially true for $T \in \mathcal{L}^1$, we can by induction assume (1.7.6) is true for decomposible tensors of degree $k-1$. Let $\ell_1 \otimes \cdots \otimes \ell_k$ be a decomposable tensor of degree k. Setting $T = \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$ we have

$$\begin{aligned} \iota_{\mathrm{v}}(\ell_1 \otimes \cdots \otimes \ell_k) &= \iota_{\mathrm{v}}(T \otimes \ell) \\ &= \iota_{\mathrm{v}} T \otimes \ell + (-1)^{k-1} \ell(v) T \end{aligned}$$

by (1.7.5). Hence

$$\begin{aligned} \iota_{\mathrm{v}}(\iota_{\mathrm{v}}(T \otimes \ell)) &= \iota_{\mathrm{v}}(\iota_{\mathrm{v}} T) \otimes \ell + (-1)^{k-2} \ell(\mathrm{v}) \iota_{\mathrm{v}} T \\ &\quad + (-1)^{k-1} \ell(v) \iota_{\mathrm{v}} T \,. \end{aligned}$$

But by induction the first summand on the right is zero and the two remaining summands cancel each other out.

□

From (1.7.6) we can deduce a slightly stronger result: For $\mathrm{v}_1, \mathrm{v}_2 \in V$

(1.7.7) $$\iota_{\mathrm{v}_1} \iota_{\mathrm{v}_2} = -\iota_{\mathrm{v}_2} \iota_{\mathrm{v}_1} \,.$$

Proof. Let $\mathrm{v} = \mathrm{v}_1 + \mathrm{v}_2$. Then $\iota_{\mathrm{v}} = \iota_{\mathrm{v}_1} + \iota_{\mathrm{v}_2}$ so

$$\begin{aligned} 0 = \iota_{\mathrm{v}} \iota_{\mathrm{v}} &= (\iota_{\mathrm{v}_1} + \iota_{\mathrm{v}_2})(\iota_{\mathrm{v}_1} + \iota_{\mathrm{v}_2}) \\ &= \iota_{\mathrm{v}_1}\iota_{\mathrm{v}_1} + \iota_{\mathrm{v}_1}\iota_{\mathrm{v}_2} + \iota_{\mathrm{v}_2}\iota_{\mathrm{v}_1} + \iota_{\mathrm{v}_2}\iota_{\mathrm{v}_2} \\ &= \iota_{\mathrm{v}_1}\iota_{\mathrm{v}_2} + \iota_{\mathrm{v}_2}\iota_{\mathrm{v}_1} \end{aligned}$$

since the first and last summands are zero by (1.7.6).

□

We'll now show how to define the operation, ι_{v}, on $\Lambda^k(V^*)$. We'll first prove

Lemma 1.7.3. *If $T \in \mathcal{L}^k$ is redundant then so is $\iota_{\mathrm{v}} T$.*

Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where ℓ is in V^*, T_1 is in $\mathcal{L}^p$ and T_2 is in $\mathcal{L}^q$. Then by (1.7.5)

$$\begin{aligned} \iota_{\mathrm{v}} T &= \iota_{\mathrm{v}} T_1 \otimes \ell \otimes \ell \otimes T_2 \\ &\quad + (-1)^p T_1 \otimes \iota_{\mathrm{v}}(\ell \otimes \ell) \otimes T_2 \\ &\quad + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \iota_{\mathrm{v}} T_2 \,. \end{aligned}$$

However, the first and the third terms on the right are redundant and

$$\iota_{\mathrm{v}}(\ell \otimes \ell) = \ell(\mathrm{v})\ell - \ell(\mathrm{v})\ell$$

by (1.7.4). □

Now let π be the projection (1.5.4) of $\mathcal{L}^k$ onto Λ^k and for $\omega = \pi(T) \in \Lambda^k$ define

$$\iota_{\mathrm{v}}\omega = \pi(\iota_{\mathrm{v}}T)\,. \tag{1.7.8}$$

To show that this definition is legitimate we note that if $\omega = \pi(T_1) = \pi(T_2)$, then $T_1 - T_2 \in \mathcal{I}^k$, so by Lemma 1.7.3 $\iota_{\mathrm{v}}T_1 - \iota_{\mathrm{v}}T_2 \in \mathcal{I}^{k-1}$ and hence

$$\pi(\iota_{\mathrm{v}}T_1) = \pi(\iota_{\mathrm{v}}T_2)\,.$$

Therefore, (1.7.8) doesn't depend on the choice of T.

By definition ι_{v} is a linear mapping of $\Lambda^k(V^*)$ into $\Lambda^{k-1}(V^*)$. We will call this the *interior product operation*. From the identities (1.7.2)–(1.7.8) one gets, for $\mathrm{v}, \mathrm{v}_1, \mathrm{v}_2 \in V$ $\omega \in \Lambda^k$, $\omega_1 \in \Lambda^p$ and $\omega_2 \in \Lambda^2$

$$\iota_{(\mathrm{v}_1+\mathrm{v}_2)}\omega = \iota_{\mathrm{v}_1}\omega + \iota_{\mathrm{v}_2}\omega \tag{1.7.9}$$

$$\iota_{\mathrm{v}}(\omega_1 \wedge \omega_2) = \iota_{\mathrm{v}}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \iota_{\mathrm{v}}\omega_2 \tag{1.7.10}$$

$$\iota_{\mathrm{v}}(\iota_{\mathrm{v}}\omega) = 0 \tag{1.7.11}$$

and

$$\iota_{\mathrm{v}_1}\iota_{\mathrm{v}_2}\omega = -\iota_{\mathrm{v}_2}\iota_{\mathrm{v}_1}\omega\,. \tag{1.7.12}$$

Moreover if $\omega = \ell_1 \wedge \cdots \wedge \ell_k$ is a decomposable element of Λ^k one gets from (1.7.4)

$$\iota_{\mathrm{v}}\omega = \sum_{r=1}^{k} (-1)^{r-1}\ell_r(\mathrm{v})\ell_1 \wedge \cdots \wedge \widehat{\ell_r} \wedge \cdots \wedge \ell_k\,. \tag{1.7.13}$$

In particular if $e_1, \ldots, e_n$ is a basis of V, $e_1^*, \ldots, e_n^*$ the dual basis of V^* and $\omega_I = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$, $1 \le i_1 < \cdots < i_k \le n$, then $\iota(e_j)\omega_I = 0$ if $j \notin I$ and if $j = i_r$

$$\iota(e_j)\omega_I = (-1)^{r-1}\omega_{I_r} \tag{1.7.14}$$

where $I_r = (i_1, \ldots, \widehat{i_r}, \ldots, i_k)$ (i.e., I_r is obtained from the multi-index I by deleting i_r).

Exercises:

1. Prove Lemma 1.7.1.

2. Prove Lemma 1.7.2.

3. Show that if $T \in \mathcal{A}^k$, $i_v = kT_v$ where T_v is the tensor (1.3.16). In particular conclude that $i_v T \in \mathcal{A}^{k-1}$. (See §1.4, exercise 8.)

4. Assume the dimension of V is n and let Ω be a non-zero element of the one dimensional vector space Λ^n. Show that the map

$$\rho : V \to \Lambda^{n-1}\,, \quad v \to \iota_v \Omega\,, \tag{1.7.15}$$

is a bijective linear map. *Hint:* One can assume $\Omega = e_1^* \wedge \cdots \wedge e_n^*$ where $e_1, \ldots, e_n$ is a basis of V. Now use (1.7.14) to compute this map on basis elements.

5. (The cross-product.) Let V be a 3-dimensional vector space, B an inner product on V and Ω a non-zero element of Λ^3. Define a map

$$V \times V \to V$$

by setting

$$v_1 \times v_2 = \rho^{-1}(Lv_1 \wedge Lv_2) \tag{1.7.16}$$

where ρ is the map (1.7.15) and $L : V \to V^*$ the map (1.2.9). Show that this map is linear in v_1, with v_2 fixed and linear in v_2 with v_1 fixed, and show that $v_1 \times v_2 = -v_2 \times v_1$.

6. For $V = \mathbb{R}^3$ let e_1, e_2 and e_3 be the standard basis vectors and B the standard inner product. (See §1.1.) Show that if $\Omega = e_1^* \wedge e_2^* \wedge e_3^*$ the cross-product above is the standard cross-product:

$$\begin{aligned} e_1 \times e_2 &= e_3 \\ e_2 \times e_3 &= e_1 \\ e_3 \times e_1 &= e_2\,. \end{aligned} \tag{1.7.17}$$

Hint: If B is the standard inner product $Le_i = e_i^*$.

Remark 1.7.4. *One can make this standard cross-product look even more standard by using the calculus notation:* $e_1 = \widehat{i}$, $e_2 = \widehat{j}$ *and* $e_3 = \widehat{k}$

1.8 The pull-back operation on Λ^k

Let V and W be vector spaces and let A be a linear map of V into W. Given a k-tensor, $T \in \mathcal{L}^k(W)$, the *pull-back*, A^*T, is the k-tensor

$$A^*T(v_1, \dots, v_k) = T(Av_1, \dots, Av_k) \tag{1.8.1}$$

in $\mathcal{L}^k(V)$. (See § 1.3, equation 1.3.12.) In this section we'll show how to define a similar pull-back operation on Λ^k.

Lemma 1.8.1. *If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.*

Proof. It suffices to verify this when T is a redundant k-tensor, i.e., a tensor of the form

$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

where $\ell_r \in W^*$ and $\ell_i = \ell_{i+1}$ for some index, i. But by (1.3.14)

$$A^*T = A^*\ell_1 \otimes \cdots \otimes A^*\ell_k$$

and the tensor on the right is redundant since $A^*\ell_i = A^*\ell_{i+1}$.

□

Now let ω be an element of $\Lambda^k(W^*)$ and let $\omega = \pi(T)$ where T is in $\mathcal{L}^k(W)$. We define

$$A^*\omega = \pi(A^*T)\,. \tag{1.8.2}$$

Claim:

The left hand side of (1.8.2) is well-defined.

Proof. If $\omega = \pi(T) = \pi(T')$, then $T = T' + S$ for some $S \in \mathcal{I}^k(W)$, and $A^*T' = A^*T + A^*S$. But $A^*S \in \mathcal{I}^k(V)$, so

$$\pi(A^*T') = \pi(A^*T)\,.$$

Proposition 1.8.2. *The map*

$$A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)\,,$$

mapping ω to $A^*\omega$ is linear. Moreover, *(i)* *If* $\omega_i \in \Lambda^{k_i}(W)$, $i = 1, 2$, *then*

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2 \,. \tag{1.8.3}$$

(ii) *If* U *is a vector space and* $B : U \to V$ *a linear map, then for* $\omega \in \Lambda^k(W^*)$,

$$B^*A^*\omega = (AB)^*\omega \,. \tag{1.8.4}$$

We'll leave the proof of these three assertions as exercises. *Hint:* They follow immediately from the analogous assertions for the pull-back operation on tensors. (See (1.3.14) and (1.3.15).)

As an application of the pull-back operation we'll show how to use it to define the notion of *determinant* for a linear mapping. Let V be a n-dimensional vector space. Then $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$; i.e., $\Lambda^n(V^*)$ is a *one-dimensional* vector space. Thus if $A : V \to V$ is a linear mapping, the induced pull-back mapping:

$$A^* : \Lambda^n(V^*) \to \Lambda^n(V^*) \,,$$

is just "multiplication by a constant". We denote this constant by $\det(A)$ and call it the *determinant* of A, Hence, by definition,

$$A^*\omega = \det(A)\omega \tag{1.8.5}$$

for all ω in $\Lambda^n(V^*)$. From (1.8.5) it's easy to derive a number of basic facts about determinants.

Proposition 1.8.3. *If* A *and* B *are linear mappings of* V *into* V, *then*

$$\det(AB) = \det(A)\det(B) \,. \tag{1.8.6}$$

Proof. By (1.8.4) and

$$\begin{aligned} (AB)^*\omega &= \det(AB)\omega \\ &= B^*(A^*\omega) = \det(B)A^*\omega \\ &= \det(B)\det(A)\omega \,, \end{aligned}$$

so, $\det(AB) = \det(A)\det(B)$.

□

Proposition 1.8.4. *If $I : V \to V$ is the identity map, $Iv = v$ for all $v \in V$,* $\det(I) = 1$.

We'll leave the proof as an exercise. *Hint:* I^* is the identity map on $\Lambda^n(V^*)$.

Proposition 1.8.5. *If $A : V \to V$ is not onto,* $\det(A) = 0$.

Proof. Let W be the image of A. Then if A is not onto, the dimension of W is less than n, so $\Lambda^n(W^*) = \{0\}$. Now let $A = I_W B$ where I_W is the inclusion map of W into V and B is the mapping, A, regarded as a mapping from V to W. Thus if ω is in $\Lambda^n(V^*)$, then by (1.8.4)

$$A^*\omega = B^* I_W^* \omega$$

and since $I_W^*\omega$ is in $\Lambda^n(W)$ it is zero.

□

□

We will derive by wedge product arguments the familiar "matrix formula" for the determinant. Let V and W be n-dimensional vector spaces and let $e_1, \dots, e_n$ be a basis for V and $f_1, \dots, f_n$ a basis for W. From these bases we get dual bases, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_n^*$, for V^* and W^*. Moreover, if A is a linear map of V into W and $[a_{i,j}]$ the $n \times n$ matrix describing A in terms of these bases, then the transpose map, $A^* : W^* \to V^*$, is described in terms of these dual bases by the $n \times n$ transpose matrix, i.e., if

$$Ae_j = \sum a_{i,j} f_i \,,$$

then

$$A^* f_j^* = \sum a_{j,i} e_i^* \,.$$

(See § 2.) Consider now $A^*(f_1^* \wedge \cdots \wedge f_n^*)$. By (1.8.3)

$$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= A^* f_1^* \wedge \cdots \wedge A^* f_n^* \\ &= \sum (a_{1,k_1} e_{k_1}^*) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^*) \end{aligned}$$

the sum being over all $k_1, \dots, k_n$, with $1 \le k_r \le n$. Thus,

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = \sum a_{1,k_1} \dots a_{n,k_n} \; e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* \,.$$

If the multi-index, $k_1, \dots, k_n$, is repeating, then $e^*_{k_1} \wedge \cdots \wedge e^*_{k_n}$ is zero, and if it's not repeating then we can write

$$k_i = \sigma(i) \quad i = 1, \dots, n$$

for some permutation, σ, and hence we can rewrite $A^*(f^*_1 \wedge \cdots \wedge f^*_n)$ as the sum over $\sigma \in S_n$ of

$$\sum a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \quad (e^*_1 \wedge \cdots \wedge e^*_n)^\sigma .$$

But

$$(e^*_1 \wedge \cdots \wedge e^*_n)^\sigma = (-1)^\sigma e^*_1 \wedge \cdots \wedge e^*_n$$

so we get finally the formula

$$A^*(f^*_1 \wedge \cdots \wedge f^*_n) = \det[a_{i,j}] e^*_1 \wedge \cdots \wedge e^*_n \tag{1.8.7}$$

where

$$\det[a_{i,j}] = \sum (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \tag{1.8.8}$$

summed over $\sigma \in S_n$. The sum on the right is (as most of you know) the *determinant* of $[a_{i,j}]$.

Notice that if $V = W$ and $e_i = f_i$, $i = 1, \dots, n$, then $\omega = e^*_1 \wedge \cdots \wedge e^*_n = f^*_1 \wedge \cdots \wedge f^*_n$, hence by (1.8.5) and (1.8.7),

$$\det(A) = \det[a_{i,j}] . \tag{1.8.9}$$

Exercises.

1. Verify the three assertions of Proposition 1.8.2.

2. Deduce from Proposition 1.8.5 a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

3. Deduce from Proposition 1.8.3 another well-known fact about determinants of $n \times n$ matrices: If one interchanges two columns, then one changes the sign of the determinant.

Hint: Let $e_1, \dots, e_n$ be a basis of V and let $B : V \to V$ be the linear mapping: $Be_i = e_j$, $Be_j = e_i$ and $Be_\ell = e_\ell$, $\ell \neq i, j$. What is $B^*(e^*_1 \wedge \cdots \wedge e^*_n)$?

4. Deduce from Propositions 1.8.3 and 1.8.4 another well-known fact about determinants of $n \times n$ matrix. If $[b_{i,j}]$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

5. Extract from (1.8.8) a well-known formula for determinants of 2×2 matrices:

$$\det \begin{bmatrix} a_{11}\,, & a_{12} \\ a_{21}\,, & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}\,.$$

6. Show that if $A = [a_{i,j}]$ is an $n \times n$ matrix and $A^t = [a_{j,i}]$ is its transpose $\det A = \det A^t$. *Hint:* You are required to show that the sums

$$\sum (-1)^\sigma a_{1,\sigma(1)} \dots a_{n,\sigma(n)} \qquad \sigma \in S_n$$

and

$$\sum (-1)^\sigma a_{\sigma(1),1} \dots a_{\sigma(n),n} \qquad \sigma \in S_n$$

are the same. Show that the second sum is identical with

$$\sum (-1)^\tau a_{\tau(1),1} \dots a_{\tau(n),n}$$

summed over $\tau = \sigma^{-1} \in S_n$.

7. Let A be an $n \times n$ matrix of the form

$$A = \begin{bmatrix} B & * \\ 0 & C \end{bmatrix}$$

where B is a $k \times k$ matrix and C the $\ell \times \ell$ matrix and the bottom $\ell \times k$ block is zero. Show that

$$\det A = \det B \det C\,.$$

Hint: Show that in (1.8.8) every non-zero term is of the form

$$(-1)^{\sigma\tau} b_{1,\sigma(1)} \dots b_{k,\sigma(k)} c_{1,\tau(1)} \dots c_{\ell,\tau(\ell)}$$

where $\sigma \in S_k$ and $\tau \in S_\ell$.

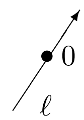
8. Let V and W be vector spaces and let $A : V \to W$ be a linear map. Show that if $Av = w$ then for $\omega \in \Lambda^p(w^*)$,

$$A^*\iota(w)\omega = \iota(v)A^*\omega\,.$$

(*Hint:* By (1.7.10) and proposition 1.8.2 it suffices to prove this for $\omega \in \Lambda^1(W^*)$, i.e., for $\omega \in W^*$.)

1.9 Orientations

We recall from freshman calculus that if $\ell \subseteq \mathbb{R}^2$ is a line through the origin, then $\ell - \{0\}$ has two connected components and an *orientation* of ℓ is a choice of one of these components (as in the figure below).



More generally, if $\mathbb{L}$ is a one-dimensional vector space then $\mathbb{L} - \{0\}$ consists of two components: namely if v is an element of $\mathbb{L} - [0\}$, then these two components are

$$\mathbb{L}_1 = \{\lambda v \ \ \lambda > 0\}$$

and

$$\mathbb{L}_2 = \{\lambda v, \ \lambda < 0\}\,.$$

An *orientation* of $\mathbb{L}$ is a choice of one of these components. Usually the component chosen is denoted $\mathbb{L}_+$, and called the *positive* component of $\mathbb{L} - \{0\}$ and the other component, $\mathbb{L}_-$, the *negative* component of $\mathbb{L} - \{0\}$.

Definition 1.9.1. *A vector, $v \in \mathbb{L}$, is positively oriented if v is in $\mathbb{L}_+$.*

More generally still let V be an n-dimensional vector space. Then $\mathbb{L} = \Lambda^n(V^*)$ is one-dimensional, and we define an *orientation* of V to be an orientation of $\mathbb{L}$. One important way of assigning an orientation to V is to choose a basis, $e_1, \dots, e_n$ of V. Then, if $e_1^*, \dots, e_n^*$ is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring that $e_1^* \wedge \cdots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$. If V has already been assigned an orientation we will say that the basis, $e_1, \dots, e_n$, is *positively* oriented if the orientation we just described coincides with the given orientation.

Suppose that $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are bases of V and that

$$e_j = \sum a_{i,j,} f_i\,. \tag{1.9.1}$$

Then by (1.7.7)

$$f_1^* \wedge \cdots \wedge f_n^* = \det[a_{i,j}] e_1^* \wedge \cdots \wedge e_n^*$$

so we conclude:

Proposition 1.9.2. *If $e_1, \ldots, e_n$ is positively oriented, then $f_1, \ldots, f_n$ is positively oriented if and only if $\det[a_{i,j}]$ is positive.*

Corollary 1.9.3. *If $e_1, \ldots, e_n$ is a positively oriented basis of V, the basis: $e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_n$ is negatively oriented.*

Now let V be a vector space of dimension $n > 1$ and W a subspace of dimension $k < n$. We will use the result above to prove the following important theorem.

Theorem 1.9.4. *Given orientations on V and V/W, one gets from these orientations a natural orientation on W.*

Remark What we mean by "natural' will be explained in the course of the proof.

Proof. Let $r = n - k$ and let π be the projection of V onto V/W . By exercises 1 and 2 of §2 we can choose a basis $e_1, \ldots, e_n$ of V such that $e_{r+1}, \ldots, e_n$ is a basis of W and $\pi(e_1), \ldots, \pi(e_r)$ a basis of V/W. Moreover, replacing e_1 by $-e_1$ if necessary we can assume by Corollary 1.9.3 that $\pi(e_1), \ldots, \pi(e_r)$ is a positively oriented basis of V/W and replacing e_n by $-e_n$ if necessary we can assume that $e_1, \ldots, e_n$ is a positively oriented basis of V. Now assign to W the orientation associated with the basis $e_{r+1}, \ldots, e_n$.

Let's show that this assignment is "natural" (i.e., doesn't depend on our choice of $e_1, \ldots, e_n$). To see this let $f_1, \ldots, f_n$ be another basis of V with the properties above and let $A = [a_{i,j}]$ be the matrix (1.9.1) expressing the vectors $e_1, \ldots, e_n$ as linear combinations of the vectors $f_1, \ldots f_n$. This matrix has to have the form

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \tag{1.9.2}$$

where B is the $r \times r$ matrix expressing the basis vectors $\pi(e_1), \ldots, \pi(e_r)$ of V/W as linear combinations of $\pi(f_1), \ldots, \pi(f_r)$ and D the $k \times k$ matrix expressing the basis vectors $e_{r+1}, \ldots, e_n$ of W as linear combinations of $f_{r+1}, \ldots, f_n$. Thus

$$\det(A) = \det(B)\det(D) .$$

However, by Proposition 1.9.2, $\det A$ and $\det B$ are positive, so $\det D$ is positive, and hence if $e_{r+1}, \dots, e_n$ is a positively oriented basis of W so is $f_{r+1}, \dots, f_n$.

□

As a special case of this theorem suppose $\dim W = n - 1$. Then the choice of a vector $v \in V - W$ gives one a basis vector, $\pi(v)$, for the one-dimensional space V/W and hence if V is oriented, the choice of v gives one a natural orientation on W.

Next let V_i, $i = 1, 2$ be oriented n-dimensional vector spaces and $A : V_1 \to V_2$ a bijective linear map. A is *orientation-preserving* if, for $\omega \in \Lambda^n(V_2^*)_+$, $A^*\omega$ is in $\Lambda^n(V_+^*)_+$. For example if $V_1 = V_2$ then $A^*\omega = \det(A)\omega$ so A is orientation preserving if and only if $\det(A) > 0$. The following proposition we'll leave as an exercise.

Proposition 1.9.5. *Let V_i, $i = 1, 2, 3$ be oriented n-dimensional vector spaces and $A_i : V_i \to V_{i+1}$, $i = 1, 2$ bijective linear maps. Then if A_1 and A_2 are orientation preserving, so is $A_2 \circ A_1$.*

Exercises.

1. Prove Corollary 1.9.3.

2. Show that the argument in the proof of Theorem 1.9.4 can be modified to prove that if V and W are oriented then these orientations induce a natural orientation on V/W.

3. Similarly show that if W and V/W are oriented these orientations induce a natural orientation on V.

4. Let V be an n-dimensional vector space and $W \subset V$ a k-dimensional subspace. Let $U = V/W$ and let $\iota : W \to V$ and $\pi : V \to U$ be the inclusion and projection maps. Suppose V and U are oriented. Let μ be in $\Lambda^{n-k}(U^*)_+$ and let ω be in $\Lambda^n(V^*)_+$. Show that there exists a ν in $\Lambda^k(V^*)$ such that $\pi^*\mu \wedge \nu = \omega$. Moreover show that $\iota^*\nu$ is *intrinsically* defined (i.e., doesn't depend on how we choose ν) and sits in the positive part, $\Lambda^k(W^*)_+$, of $\Lambda^k(W)$.

5. Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{R}^n$. The *standard* orientation of $\mathbb{R}^n$ is, by definition, the orientation associated with this basis. Show that if W is the subspace of $\mathbb{R}^n$ defined by the

equation, $x_1 = 0$, and $v = e_1 \notin W$ then the natural orientation of W associated with v and the standard orientation of $\mathbb{R}^n$ coincide with the orientation given by the basis vectors, $e_2, \dots, e_n$ of W.

6. Let V be an oriented n-dimensional vector space and W an $n-1$-dimensional subspace. Show that if v and v' are in $V - W$ then $v' = \lambda v + w$, where w is in W and $\lambda \in \mathbb{R} - \{0\}$. Show that v and v' give rise to the same orientation of W if and only if λ is positive.

7. Prove Proposition 1.9.5.

8. A key step in the proof of Theorem 1.9.4 was the assertion that the matrix A expressing the vectors, e_i, as linear combinations of the vectors, f_i, had to have the form (1.9.2). Why is this the case?

9. (a) Let V be a vector space, W a subspace of V and $A : V \to V$ a bijective linear map which maps W onto W. Show that one gets from A a bijective linear map

$$B : V/W \to V/W$$

with property

$$\pi A = B\pi ,$$

π being the projection of V onto V/W.

(b) Assume that V, W and V/W are compatibly oriented. Show that if A is orientation-preserving and its restriction to W is orientation preserving then B is orientation preserving.

10. Let V be a oriented n-dimensional vector space, W an $(n-1)$-dimensional subspace of V and $i : W \to V$ the inclusion map. Given $\omega \in \Lambda^b(V)_+$ and $v \in V - W$ show that for the orientation of W described in exercise 5, $i^*(\iota_v \omega) \in \Lambda^{n-1}(W)_+$.

11. Let V be an n-dimensional vector space, $B : V \times V \to \mathbb{R}$ an inner product and $e_1, \dots, e_n$ a basis of V which is positively oriented and orthonormal. Show that the "volume element"

$$\text{vol} = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)$$

is intrinsically defined, independent of the choice of this basis. *Hint:* (1.2.13) and (1.8.7).

12. (a) Let V be an oriented n-dimensional vector space and B an inner product on V. Fix an oriented orthonormal basis, $e_1, \dots, e_n$, of V and let $A : V \to V$ be a linear map. Show that if

$$Ae_i = \mathrm{v}_i = \sum a_{j,i} e_j$$

and $b_{i,j} = B(\mathrm{v}_i, \mathrm{v}_j)$, the matrices $\mathcal{A} = [a_{i,j}]$ and $\mathcal{B} = [b_{i,j}]$ are related by: $\mathcal{B} = \mathcal{A}^+ \mathcal{A}$.

(b) Show that if ν is the volume form, $e_1^* \wedge \cdots \wedge e_n^*$, and A is orientation preserving

$$A^* \nu = (\det \mathcal{B})^{\frac{1}{2}} \nu\,.$$

(c) By Theorem 1.5.6 one has a bijective map

$$\Lambda^n(V^*) \cong \mathcal{A}^n(V)\,.$$

Show that the element, Ω, of $\mathcal{A}^n(V)$ corresponding to the form, ν, has the property

$$|\Omega(\mathrm{v}_1, \dots, \mathrm{v}_n)|^2 = \det([b_{i,j}])$$

where $\mathrm{v}_1, \dots, \mathrm{v}_n$ are any n-tuple of vectors in V and $b_{i,j} = B(\mathrm{v}_i, \mathrm{v}_j)$.

CHAPTER 2

DIFFERENTIAL FORMS

2.1 Vector fields and one-forms

The goal of this chapter is to generalize to n dimensions the basic operations of three dimensional vector calculus: div, curl and grad. The "div", and "grad" operations have fairly straight forward generalizations, but the "curl" operation is more subtle. For vector fields it doesn't have any obvious generalization, however, if one replaces vector fields by a closely related class of objects, differential forms, then not only does it have a natural generalization but it turns out that div, curl and grad are all special cases of a general operation on differential forms called *exterior differentiation*.

In this section we will review some basic facts about vector fields in n variables and introduce their dual objects: *one-forms*. We will then take up in §2.2 the theory of k-forms for k greater than one. We begin by fixing some notation.

Given $p \in \mathbb{R}^n$ we define the tangent space to $\mathbb{R}^n$ at p to be the set of pairs

$$T_p\mathbb{R}^n = \{(p, \mathrm{v})\}\,; \quad \mathrm{v} \in \mathbb{R}^n\,. \tag{2.1.1}$$

The identification

$$T_p\mathbb{R}^n \to \mathbb{R}^n\,, \quad (p, \mathrm{v}) \to \mathrm{v} \tag{2.1.2}$$

makes $T_p\mathbb{R}^n$ into a vector space. More explicitly, for v, v_1 and $\mathrm{v}_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define the addition and scalar multiplication operations on $T_p\mathbb{R}^n$ by the recipes

$$(p, \mathrm{v}_1) + (p, \mathrm{v}_2) \quad = \quad (p, \mathrm{v}_1 + \mathrm{v}_2)$$

and

$$\lambda(p, \mathrm{v}) \quad = \quad (p, \lambda \mathrm{v})\,.$$

Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a C^1 map. We recall that the derivative

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^m$$

of f at p is the linear map associated with the $m \times n$ matrix

$$\left[\frac{\partial f_i}{\partial x_j}(p)\right] .$$

It will be useful to have a "base-pointed" version of this definition as well. Namely, if $q = f(p)$ we will define

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

to be the map

$$df_p(p, \mathrm{v}) = (q, Df(p)\mathrm{v}) . \tag{2.1.3}$$

It's clear from the way we've defined vector space structures on $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^m$ that this map is linear.

Suppose that the image of f is contained in an open set, V, and suppose $g : V \to \mathbb{R}^k$ is a C^1 map. Then the "base-pointed"" version of the chain rule asserts that

$$dg_q \circ df_p = d(f \circ g)_p . \tag{2.1.4}$$

(This is just an alternative way of writing $Dg(q)Df(p) = D(g \circ f)(p)$.)

In 3-dimensional vector calculus a *vector field* is a function which attaches to each point, p, of $\mathbb{R}^3$ a base-pointed arrow, $(p, \vec{\mathrm{v}})$. The n-dimensional version of this definition is essentially the same.

Definition 2.1.1. *Let U be an open subset of $\mathbb{R}^n$. A vector field on U is a function, v, which assigns to each point, p, of U a vector $v(p)$ in $T_p\mathbb{R}^n$.*

Thus a vector field is a vector-valued function, but its value at p is an element of a vector space, $T_p\mathbb{R}^n$ that itself depends on p.

Some examples.

1. Given a fixed vector, $\mathrm{v} \in \mathbb{R}^n$, the function

$$p \in \mathbb{R}^n \to (p, \mathrm{v}) \tag{2.1.5}$$

is a vector field. Vector fields of this type are *constant* vector fields.

2. In particular let $e_i, i = 1, \ldots, n$, be the standard basis vectors of $\mathbb{R}^n$. If $\mathrm{v} = e_i$ we will denote the vector field (2.1.5) by $\partial/\partial x_i$. (The reason for this "derivation notation" will be explained below.)

3. Given a vector field on U and a function, $f : U \to \mathbb{R}$ we'll denote by fv the vector field

$$p \in U \to f(p)v(p)\,.$$

4. Given vector fields v_1 and v_2 on U, we'll denote by $v_1 + v_2$ the vector field

$$p \in U \to v_1(p) + v_2(p)\,.$$

5. The vectors, (p, e_i), $i = 1, \ldots, n$, are a basis of $T_p\mathbb{R}^n$, so if v is a vector field on U, $v(p)$ can be written uniquely as a linear combination of these vectors with real numbers, $g_i(p)$, $i = 1, \ldots, n$, as coefficients. In other words, using the notation in example 2 above, v can be written uniquely as a sum

$$v = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \tag{2.1.6}$$

where $g_i : U \to \mathbb{R}$ is the function, $p \to g_i(p)$.

We'll say that v is a $\mathcal{C}^\infty$ vector field if the g_i's are in $\mathcal{C}^\infty(U)$.

A basic vector field operation is *Lie differentiation.* If $f \in C^1(U)$ we define $L_v f$ to be the function on U whose value at p is given by

$$Df(p)\mathrm{v} = L_v f(p) \tag{2.1.7}$$

where $v(p) = (p, \mathrm{v})$. If v is the vector field (2.1.6) then

$$L_v f = \sum g_i \frac{\partial}{\partial x_i} f \tag{2.1.8}$$

(motivating our "derivation notation" for v).

Exercise.

Check that if $f_i \in C^1(U)$, $i = 1, 2$, then

$$L_v(f_1 f_2) = f_1 L_v f_2 + f_1 L_v f_2\,. \tag{2.1.9}$$

Next we'll generalize to n-variables the calculus notion of an "integral curve" of a vector field.

Definition 2.1.2. *A C^1 curve $\gamma : (a,b) \to U$ is an integral curve of v if for all $a < t < b$ and $p = \gamma(t)$*

$$\left(p, \frac{d\gamma}{dt}(t)\right) = v(p)$$

i.e., if v is the vector field (2.1.6) and $g : U \to \mathbb{R}^n$ is the function $(g_1, \ldots, g_n)$ the condition for $\gamma(t)$ to be an integral curve of v is that it satisfy the system of differential equations

$$\frac{d\gamma}{dt}(t) = g(\gamma(t))\,. \tag{2.1.10}$$

We will quote without proof a number of basic facts about systems of ordinary differential equations of the type (2.1.10). (A source for these results that we highly recommend is Birkhoff–Rota, *Ordinary Differential Equations*, Chapter 6.)

Theorem 2.1.3 (Existence)**.** *Given a point $p_0 \in U$ and $a \in \mathbb{R}$, there exists an interval $I = (a - T, a + T)$, a neighborhood, U_0, of p_0 in U and for every $p \in U_0$ an integral curve, $\gamma_p : I \to U$ with $\gamma_p(a) = p$.*

Theorem 2.1.4 (Uniqueness)**.** *Let $\gamma_i : I_i \to U$, $i = 1, 2$, be integral curves. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2 \to U$ defined by*

$$\gamma(t) = \begin{cases} \gamma_1(t)\,, & t \in I_1 \\ \gamma_2(t)\,, & t \in I_2 \end{cases}$$

is an integral curve.

Theorem 2.1.5 (Smooth dependence on initial data)**.** *Let v be a $\mathcal{C}^\infty$-vector field, on an open subset, V, of U, $I \subseteq \mathbb{R}$ an open interval, $a \in I$ a point on this interval and $h : V \times I \to U$ a mapping with the properties:*

> *(i) $h(p, a) = p$.*
>
> *(ii) For all $p \in V$ the curve*
>
> $$\gamma_p : I \to U \qquad \gamma_p(t) = h(p,t)$$
>
> *is an integral curve of v.*
>
> *Then the mapping, h, is $\mathcal{C}^\infty$.*

One important feature of the system (2.1.11) is that it is an *autonomous* system of differential equations: the function, $g(x)$, is a function of x alone, it doesn't depend on t. One consequence of this is the following:

Theorem 2.1.6. *Let $I = (a, b)$ and for $c \in \mathbb{R}$ let $I_c = (a - c, b - c)$. Then if $\gamma : I \to U$ is an integral curve, the reparametrized curve*

$$\gamma_c : I_c \to U\,, \quad \gamma_c(t) = \gamma(t + c) \tag{2.1.11}$$

is an integral curve.

We recall that a C^1-function $\varphi : U \to \mathbb{R}$ is an *integral* of the system (2.1.11) if for every integral curve $\gamma(t)$, the function $t \to \varphi(\gamma(t))$ is constant. This is true if and only if for all t and $p = \gamma(t)$

$$0 = \frac{d}{dt}\varphi(\gamma(t)) = (D\varphi)_p \left(\frac{d\gamma}{dt}\right) = (D\varphi)_p(\mathrm{v})$$

where $(p, \mathrm{v}) = v(p)$. But by (2.1.6) the term on the right is $L_v\varphi(p)$. Hence we conclude

Theorem 2.1.7. *$\varphi \in C^1(U)$ is an integral of the system (2.1.11) if and only if $L_v\varphi = 0$.*

We'll now discuss a class of objects which are in some sense "dual objects" to vector fields. For each $p \in \mathbb{R}^n$ let $(T_p\mathbb{R})^*$ be the dual vector space to $T_p\mathbb{R}^n$, i.e., the space of all linear mappings, $\ell : T_p\mathbb{R}^n \to \mathbb{R}$.

Definition 2.1.8. *Let U be an open subset of $\mathbb{R}^n$. A* one-form *on U is a function, ω, which assigns to each point, p, of U a vector, ω_p, in $(T_p\mathbb{R}^n)^*$.*

Some examples:

1. Let $f : U \to \mathbb{R}$ be a C^1 function. Then for $p \in U$ and $c = f(p)$ one has a linear map

$$df_p : T_p\mathbb{R}^n \to T_c\mathbb{R} \tag{2.1.12}$$

and by making the identification,

$$T_c\mathbb{R} = \{c, \mathbb{R}\} = \mathbb{R}$$

df_p can be regarded as a linear map from $T_p\mathbb{R}^n$ to $\mathbb{R}$, i.e., as an element of $(T_p\mathbb{R}^n)^*$. Hence the assignment

$$p \in U \to df_p \in (T_p\mathbb{R}^n)^* \tag{2.1.13}$$

defines a one-form on U which we'll denote by df.

2. Given a one-form ω and a function, $\varphi : U \to \mathbb{R}$ the product of φ with ω is the one-form, $p \in U \to \varphi(p)\omega_p$.

3. Given two one-forms ω_1 and ω_2 their sum, $\omega_1 + \omega_2$ is the one-form, $p \in U \to \omega_1(p) + \omega_2(p)$.

4. The one-forms $dx_1, \ldots, dx_n$ play a particularly important role. By (2.1.12)

$$(dx_i)\left(\frac{\partial}{\partial x_j}\right)_p = \delta_{ij} \tag{2.1.14}$$

i.e., is equal to 1 if $i = j$ and zero if $i \neq j$. Thus $(dx_1)_p, \ldots, (dx_n)_p$ are the basis of $(T_p^*\mathbb{R}^n)^*$ *dual* to the basis $(\partial/\partial x_i)_p$. Therefore, if ω is *any* one-form on U, ω_p can be written uniquely as a sum

$$\omega_p = \sum f_i(p)(dx_i)_p\,, \quad f_i(p) \in \mathbb{R}$$

and ω can be written uniquely as a sum

$$\omega = \sum f_i\, dx_i \tag{2.1.15}$$

where $f_i : U \to \mathbb{R}$ is the function, $p \to f_i(p)$. We'll say that ω *is a* $\mathcal{C}^\infty$ *one-form* if the f_i's are $\mathcal{C}^\infty$.

Exercise.

Check that if $f : U \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function

$$df = \sum \frac{\partial f}{\partial x_i}\, dx_i\,. \tag{2.1.16}$$

Suppose now that v is a vector field and ω a one-form on U. Then for every $p \in U$ the vectors, $v_p \in T_p\mathbb{R}^n$ and $\omega_p \in (T_p\mathbb{R}^n)^*$ can be paired to give a number, $\iota(v_p)\omega_p \in \mathbb{R}$, and hence, as p varies, an

$\mathbb{R}$-valued function, $\iota(v)\omega$, which we will call the *interior product* of v with ω. For instance if v is the vector field (2.1.6) and ω the one-form (2.1.15) then

$$\iota(v)\omega = \sum f_i g_i \,. \tag{2.1.17}$$

Thus if v and ω are $\mathcal{C}^\infty$ so is the function $\iota(v)\omega$. Also notice that if $\varphi \in \mathcal{C}^\infty(U)$, then as we observed above

$$d\varphi = \sum \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_i}$$

so if v is the vector field (2.1.6)

$$\iota(v)\, d\varphi = \sum g_i \frac{\partial \varphi}{\partial x_i} = L_v \varphi \,. \tag{2.1.18}$$

Coming back to the theory of integral curves, let U be an open subset of $\mathbb{R}^n$ and v a vector field on U. We'll say that v is *complete* if, for every $p \in U$, there exists an integral curve, $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$, i.e., for every p there exists an integral curve that starts at p and *exists for all time.* To see what "completeness" involves, we recall that an integral curve

$$\gamma : [0, b) \to U \,,$$

with $\gamma(0) = p$, is called *maximal* if it can't be extended to an interval $[0, b')$, $b' > b$. (See for instance Birkhoff–Rota, §6.11.) For such curves it's known that either

i. $b = +\infty$
or

ii. $|\gamma(t)| \to +\infty$ as $t \to b$
or

iii. the limit set of

$$\{\gamma(t) \,, \quad 0 \leq t, b\}$$

contains points on the boundary of U.

Hence if we can exclude ii. and iii. we'll have shown that an integral curve with $\gamma(0) = p$ exists for all positive time. A simple criterion for excluding ii. and iii. is the following.

Lemma 2.1.9. *The scenarios ii. and iii. can't happen if there exists a proper C^1-function, $\varphi : U \to \mathbb{R}$ with $L_v\varphi = 0$.*

Proof. $L_v\varphi = 0$ implies that φ is constant on $\gamma(t)$, but if $\varphi(p) = c$ this implies that the curve, $\gamma(t)$, lies on the compact subset, $\varphi^{-1}(c)$, of U; hence it can't "run off to infinity" as in scenario ii. or "run off the boundary" as in scenario iii.

□

Applying a similar argument to the interval $(-b, 0]$ we conclude:

Theorem 2.1.10. *Suppose there exists a proper C^1-function, $\varphi : U \to \mathbb{R}$ with the property $L_v\varphi = 0$. Then v is complete.*

Example.

Let $U = \mathbb{R}^2$ and let v be the vector field

$$v = x^3 \frac{\partial}{\partial y} - y\frac{\partial}{\partial x} .$$

Then $\varphi(x, y) = 2y^2 + x^4$ is a proper function with the property above. Another hypothesis on v which excludes ii. and iii. is the following. We'll define the *support* of v to be the set

$$\operatorname{supp} v = \overline{q \in U\,, \quad v(q) \neq 0\}}\,,$$

and will say that v is compactly supported if this set is compact. We will prove

Theorem 2.1.11. *If v is compactly supported it is complete.*

Proof. Notice first that if $v(p) = 0$, the constant curve, $\gamma_0(t) = p$, $-\infty < t < \infty$, satisfies the equation

$$\frac{d}{dt}\gamma_0(t) = 0 = v(p)\,,$$

so it is an integral curve of v. Hence if $\gamma(t)$, $-a < t < b$, is any integral curve of v with the property, $\gamma(t_0) = p$, for some t_0, it has to coincide with γ_0 on the interval, $-a < t < a$, and hence has to be the constant curve, $\gamma(t) = p$, on this interval.

Now suppose the support, A, of v is compact. Then either $\gamma(t)$ is in A for all t or is in $U - A$ for some t_0. But if this happens, and

$p = \gamma(t_0)$ then $v(p) = 0$, so $\gamma(t)$ has to coincide with the constant curve, $\gamma_0(t) = p$, for all t. In neither case can it go off to ∞ or off to the boundary of U as $t \to b$.

□

One useful application of this result is the following. Suppose v is a vector field on U, and one wants to see what its integral curves look like on some compact set, $A \subseteq U$. Let $\rho \in \mathcal{C}_0^\infty(U)$ be a bump function which is equal to one on a neighborhood of A. Then the vector field, $w = \rho v$, is compactly supported and hence complete, but it is identical with v on A, so its integral curves on A coincide with the integral curves of v.

If v is complete then for every p, one has an integral curve, $\gamma_p : \mathbb{R} \to U$ with $\gamma_p(0) = p$, so one can define a map

$$f_t : U \to U$$

by setting $f_t(p) = \gamma_p(t)$. If v is $\mathcal{C}^\infty$, this mapping is $\mathcal{C}^\infty$ by the smooth dependence on initial data theorem, and by definition f_0 is the identity map, i.e., $f_0(p) = \gamma_p(0) = p$. We claim that the f_t's also have the property

$$f_t \circ f_a = f_{t+a} \,. \tag{2.1.19}$$

Indeed if $f_a(p) = q$, then by the reparametrization theorem, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of v, and since $q = \gamma_q(0) = \gamma_p(a) = f_a(p)$, they have the same initial point, so

$$\begin{aligned} \gamma_q(t) &= f_t(q) = (f_t \circ f_a)(p) \\ &= \gamma_p(t+a) = f_{t+a}(p) \end{aligned}$$

for all t. Since f_0 is the identity it follows from (2.1.19) that $f_t \circ f_{-t}$ is the identity, i.e.,

$$f_{-t} = f_t^{-1} \,,$$

so f_t is a $\mathcal{C}^\infty$ diffeomorphism. Hence if v is complete it generates a "one-parameter group", f_t, $-\infty < t < \infty$, of $\mathcal{C}^\infty$-diffeomorphisms.

For v not complete there is an analogous result, but it's trickier to formulate precisely. Roughly speaking v generates a one-parameter group of diffeomorphisms, f_t, but these diffeomorphisms are not defined on all of U nor for all values of t. Moreover, the identity (2.1.19) only holds on the open subset of U where both sides are well-defined.

We'll devote the remainder of this section to discussing some "functorial" properties of vector fields and one-forms. Let U and W be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : U \to W$ be a $\mathcal{C}^\infty$ map. If v is a $\mathcal{C}^\infty$-vector field on U and w a $\mathcal{C}^\infty$-vector field on W we will say that v and w are "f-related" if, for all $p \in U$ and $q = f(p)$

$$df_p(v_p) = \mathrm{w}_q\,. \tag{2.1.20}$$

Writing

$$v \quad = \quad \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\,, \qquad v_i \in C^k(U)$$

and

$$\mathrm{w} \quad = \quad \sum_{j=1}^{m} \mathrm{w}_j \frac{\partial}{\partial y_j}\,, \qquad \mathrm{w}_j \in C^k(V)$$

this equation reduces, in coordinates, to the equation

$$\mathrm{w}_i(q) = \sum \frac{\partial f_i}{\partial x_j}(p) v_j(p)\,. \tag{2.1.21}$$

In particular, if $m = n$ and f is a $\mathcal{C}^\infty$ diffeomorphism, the formula (3.2) defines a $\mathcal{C}^\infty$-vector field on W, i.e.,

$$\mathrm{w} = \sum_{j=1}^{n} \mathrm{w}_i \frac{\partial}{\partial y_j}$$

is the vector field defined by the equation

$$\mathrm{w}_i = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}\,. \tag{2.1.22}$$

Hence we've proved

Theorem 2.1.12. *If $f : U \to W$ is a $\mathcal{C}^\infty$ diffeomorphism and v a $\mathcal{C}^\infty$-vector field on U, there exists a unique $\mathcal{C}^\infty$ vector field, w, on W having the property that v and w are f-related.*

We'll denote this vector field by $f_* v$ and call it the *push-forward of v by f*.

I'll leave the following assertions as easy exercises.

Theorem 2.1.13. *Let U_i, $i = 1, 2$, be open subsets of $\mathbb{R}^{n_i}$, v_i a vector field on U_i and $f : U_1 \to U_2$ a $\mathcal{C}^\infty$-map. If v_1 and v_2 are f-related, every integral curve*

$$\gamma : I \to U_1$$

of v_1 gets mapped by f onto an integral curve, $f \circ \gamma : I \to U_2$, of v_2.

Corollary 2.1.14. *Suppose v_1 and v_2 are complete. Let $(f_i)_t : U_i \to U_i$, $-\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v_i. Then $f \circ (f_1)_t = (f_2)_t \circ f$.*

Hints:

1. Theorem 4 follows from the chain rule: If $p = \gamma(t)$ and $q = f(p)$

$$df_p \left(\frac{d}{dt} \gamma(t) \right) = \frac{d}{dt} f(\gamma(t)) \,.$$

2. To deduce Corollary 5 from Theorem 4 note that for $p \in U$, $(f_1)_t(p)$ is just the integral curve, $\gamma_p(t)$ of v_1 with initial point $\gamma_p(0) = p$.

The notion of f-relatedness can be very succinctly expressed in terms of the Lie differentiation operation. For $\varphi \in \mathcal{C}^\infty(U_2)$ let $f^*\varphi$ be the composition, $\varphi \circ f$, viewed as a $\mathcal{C}^\infty$ function on U_1, i.e., for $p \in U_1$ let $f^*\varphi(p) = \varphi(f(p))$. Then

$$f^* L_{v_2} \varphi = L_{v_1} f^* \varphi \,. \tag{2.1.23}$$

(To see this note that if $f(p) = q$ then at the point p the right hand side is

$$(d\varphi)_q \circ df_p(v_1(p))$$

by the chain rule and by definition the left hand side is

$$d\varphi_q(v_2(q)) \,.$$

Moreover, by definition

$$v_2(q) = df_p(v_1(p))$$

so the two sides are the same.)

Another easy consequence of the chain rule is:

Theorem 2.1.15. *Let U_i, $i = 1, 2, 3$, be open subsets of $\mathbb{R}^{n_i}$, v_i a vector field on U_i and $f_i : U_i \to U_{i+1}$, $i = 1, 2$ a $\mathcal{C}^\infty$-map. Suppose that, for $i = 1, 2$, v_i and v_{i+1} are f_i-related. Then v_1 and v_3 are $f_2 \circ f_1$-related.*

In particular, if f_1 and f_2 are diffeomorphisms and $v = v_1$

$$(f_2)_*(f_1)_*v = (f_2 \circ f_1)_*v\,.$$

The results we described above have "dual" analogues for one-forms. Namely, let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : U \to V$ be a $\mathcal{C}^\infty$-map. Given a one-form, μ, on V one can define a "pull-back" one-form, $f^*\mu$, on U by the following method. For $p \in U$ let $q = f(p)$. By definition $\mu(q)$ is a linear map

$$\mu(q) : T_q\mathbb{R}^m \to \mathbb{R} \tag{2.1.24}$$

and by composing this map with the linear map

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^n$$

we get a linear map

$$\mu_q \circ df_p : T_p\mathbb{R}^n \to \mathbb{R}\,,$$

i.e., an element $\mu_q \circ df_p$ of $T_p^*\mathbb{R}^n$.

Definition 2.1.16. *The one-form $f^*\mu$ is the one-form defined by the map*

$$p \in U \to (\mu_q \circ df_p) \in T_p^*\mathbb{R}^n$$

where $q = f(p)$.

Note that if $\varphi : V \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function and $\mu = d\varphi$ then

$$\mu_q \circ df_p = d\varphi_q \circ df_p = d(\varphi \circ f)_p$$

i.e.,

$$f^*\mu = d\varphi \circ f\,. \tag{2.1.25}$$

In particular if μ is a one-form of the form, $\mu = d\varphi$, with $\varphi \in \mathcal{C}^\infty(V)$, $f^*\mu$ is $\mathcal{C}^\infty$. From this it is easy to deduce

Theorem 2.1.17. *If μ is any $\mathcal{C}^\infty$ one-form on V, its pull-back, $f^*\omega$, is $\mathcal{C}^\infty$. (See exercise 1.)*

Notice also that the pull-back operation on one-forms and the push-forward operation on vector fields are somewhat different in character. The former is defined for *all* $\mathcal{C}^\infty$ maps, but the latter is only defined for diffeomorphisms.

Exercises.

1. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^n$ and $f : U \to V$ a C^k map.

(a) Show that for $\varphi \in \mathcal{C}^\infty(V)$ (2.1.25) can be rewritten

$$f^* \, d\varphi = df^* \varphi \,. \tag{2.1.25$'$}$$

(b) Let μ be the one-form

$$\mu = \sum_{i=1}^{m} \varphi_i \, dx_i \qquad \varphi_i \in \mathcal{C}^\infty(V)$$

on V. Show that if $f = (f_1, \ldots, f_m)$ then

$$f^* \mu = \sum_{i=1}^{m} f^* \varphi_i \, df_i \,.$$

(c) Show that if μ is $\mathcal{C}^\infty$ and f is $\mathcal{C}^\infty$, $f^*\mu$ is $\mathcal{C}^\infty$.

2. Let v be a complete vector field on U and $f_t : U \to U$, the one parameter group of diffeomorphisms generated by v. Show that if $\varphi \in C^1(U)$

$$L_v \varphi = \left(\frac{d}{dt} f_t^* \varphi \right)_{t=0} \,.$$

3. (a) Let $U = \mathbb{R}^2$ and let $\mathfrak{v}$ be the vector field, $x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1$. Show that the curve

$$t \in \mathbb{R} \to (r \cos(t + \theta) \,, \, r \sin(t + \theta))$$

is the unique integral curve of $\mathfrak{v}$ passing through the point, $(r \cos\theta, r \sin\theta)$, at $t = 0$.

(b) Let $U = \mathbb{R}^n$ and let $\mathfrak{v}$ be the constant vector field: $\sum c_i \partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to a + t(c_1, \ldots, c_n)$$

is the unique integral curve of $\mathfrak{v}$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

(c) Let $U = \mathbb{R}^n$ and let $\mathfrak{v}$ be the vector field, $\sum x_i \partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to e^t(a_1, \ldots, a_n)$$

is the unique integral curve of $\mathfrak{v}$ passing through a at $t = 0$.

4. Show that the following are one-parameter groups of diffeomorphisms:

(a) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = x + t$

(b) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = e^t x$

(c) $f_t : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad f_t(x, y) = (\cos t\, x - \sin t\, y\,,\, \sin t\, x + \cos t\, y)$

5. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Show that the series

$$\exp tA = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbb{R}^n$.

6. (a) What are the infinitesimal generators of the one-parameter groups in exercise 13?

(b) Show that the infinitesimal generator of the one-parameter group in exercise 14 is the vector field

$$\sum a_{i,j} x_j \frac{\partial}{\partial x_i}$$

where $[a_{i,j}]$ is the defining matrix of A.

7. Let v be the vector field on $\mathbb{R}$, $x^2 \frac{d}{dx}$ Show that the curve

$$x(t) = \frac{a}{a - at}$$

is an integral curve of v with initial point $x(0) = a$. Conclude that for $a > 0$ the curve

$$x(t) = \frac{a}{1 - at}\,, \quad 0 < t < \frac{1}{a}$$

is a maximal integral curve. (In particular, conclude that v isn't complete.)

8. Let U be an open subset of $\mathbb{R}^n$ and v_1 and v_2 vector fields on U. Show that there is a unique vector field, w, on U with the property

$$L_w \varphi = L_{v_1}(L_{v_2}\varphi) - L_{v_2}(L_{v_1}\varphi)$$

for all $\varphi \in \mathcal{C}^\infty(U)$.

9. The vector field w in exercise 8 is called the *Lie bracket* of the vector fields v_1 and v_2 and is denoted $[v_1, v_2]$. Verify that "Lie bracket" satisfies the identities

$$[v_1, v_2] = -[v_2, v_1]$$

and

$$[v_1[v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0\,.$$

Hint: Prove analogous identities for L_{v_1}, L_{v_2} and L_{v_3}.

10. Let $v_1 = \partial/\partial x_i$ and $v_2 = \sum g_j \partial/\partial x_j$. Show that

$$[v_1, v_2] = \sum \frac{\partial}{\partial x_i} g_i \frac{\partial}{\partial x_j}\,.$$

11. Let v_1 and v_2 be vector fields and f a $\mathcal{C}^\infty$ function. Show that

$$[v_1, f v_2] = L_{v_1} f v_2 + f[v_1, v_2]\,.$$

12. Let U and V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ a diffeomorphism. If w is a vector field on V, define the pull-back, $f^* w$ of w to U to be the vector field

$$f^* w = (f_*^{-1} w)\,.$$

Show that if φ is a $\mathcal{C}^\infty$ function on V

$$f^* L_w \varphi = L_{f^* w} f^* \varphi\,.$$

Hint: (2.1.26).

13. Let U be an open subset of $\mathbb{R}^n$ and v and w vector fields on U. Suppose v is the infinitesimal generator of a one-parameter group of diffeomorphisms

$$f_t : U \to U\,, \quad -\infty < t < \infty\,.$$

Let $w_t = f_t^* w$. Show that for $\varphi \in \mathcal{C}^\infty(U)$

$$L_{[v,w]}\varphi \quad = \quad L_{\dot{w}}\varphi$$

where

$$\dot{w} \quad = \quad \frac{d}{dt} f_t^* w \,|_{t=0}\,.$$

Hint: Differentiate the identity

$$f_t^* L_w \varphi = L_{w_t} f_t^* \varphi$$

with respect to t and show that at $t = 0$ the derivative of the left hand side is

$$L_v L_w \varphi$$

by exercise 2 and the derivative of the right hand side is

$$L_{\dot{w}} + L_w(L_v\varphi)\,.$$

14. Conclude from exercise 13 that

$$[v,w] = \frac{d}{dt} f_t^* w \,|_{t=0}\,. \tag{2.1.26}$$

15. Let U be an open subset of $\mathbb{R}^n$ and let $\gamma : [a,b] \to U$, $t \to (\gamma_1(t), \dots, \gamma_n(t))$ be a C^1 curve. Given $\omega = \sum f_i\, dx_i \in \Omega^1(U)$, define the *line integral* of ω over γ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt}\, dt\,.$$

Show that if $\omega = df$ for some $f \in \mathcal{C}^\infty(U)$

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))\,.$$

In particular conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

16. Let

$$\omega = \frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 - \{0\})\,,$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ be the closed curve, $t \to (\cos t, \sin t)$. Compute the line integral, $\int_\gamma \omega$, and show that it's not zero. Conclude that ω can't be "d" of a function, $f \in \mathcal{C}^\infty(\mathbb{R}^2 - \{0\})$.

17. Let f be the function

$$f(x_1, x_2) = \begin{cases} \arctan \frac{x_2}{x_1}\,, x_1 > 0 \\ \frac{\pi}{2}\,,\ x_1 = 0\,,\ x_2 > 0 \\ \arctan \frac{x_2}{x_1} + \pi\,,\ x_1 < 0 \end{cases}$$

where, we recall: $-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}$. Show that this function is $\mathcal{C}^\infty$ and that df is the 1-form, ω, in the previous exercise. Why doesn't this contradict what you proved in exercise 16?

2.2 k-forms

One-forms are the bottom tier in a pyramid of objects whose k^{th} tier is the space of k-*forms*. More explicitly, given $p \in \mathbb{R}^n$ we can, as in §1.5, form the k^{th} exterior powers

$$\Lambda^k(T_p^*\mathbb{R}^n)\,, \quad k = 1, 2, 3, \ldots, n \tag{2.2.1}$$

of the vector space, $T_p^*\mathbb{R}^n$, and since

$$\Lambda^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n \tag{2.2.2}$$

one can think of a one-form as a function which takes its value at p in the space (2.2.2). This leads to an obvious generalization.

Definition 2.2.1. *Let U be an open subset of $\mathbb{R}^n$. A k-form, ω, on U is a function which assigns to each point, p, in U an element $\omega(p)$ of the space (2.2.1) .*

The wedge product operation gives us a way to construct lots of examples of such objects.

Example 1.

Let ω_i, $i = 1, \dots, k$ be one-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product

$$\omega_1(p) \wedge \cdots \wedge \omega_k(p) \,. \tag{2.2.3}$$

Notice that since $\omega_i(p)$ is in $\Lambda^1(T_p^*\mathbb{R}^n)$ the wedge product (2.2.3) makes sense and is an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.

Example 2.

Let f_i, $i = 1, \dots, k$ be a real-valued $\mathcal{C}^\infty$ function on U. Letting $\omega_i = df_i$ we get from (2.2.3) a k-form

$$df_1 \wedge \cdots \wedge df_k \tag{2.2.4}$$

whose value at p is the wedge product

$$(df_1)_p \wedge \cdots \wedge (df_k)_p \,. \tag{2.2.5}$$

Since $(dx_1)_p, \dots, (dx_n)_p$ are a basis of $T_p^*\mathbb{R}^n$, the wedge products

$$(dx_{i_1})_p \wedge \cdots \wedge (dx_{1_k})_p \,, \quad 1 \leq i_1 < \cdots < i_k \leq n \tag{2.2.6}$$

are a basis of $\Lambda^k(T_p^*)$. To keep our multi-index notation from getting out of hand, we'll denote these basis vectors by $(dx_I)_p$, where $I = (i_1, \dots, i_k)$ and the I's range over multi-indices of length k which are *strictly increasing*. Since these wedge products are a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$ every element of $\Lambda^k(T_p^*\mathbb{R}^n)$ can be written uniquely as a sum

$$\sum c_I (dx_I)_p \,, \quad c_I \in \mathbb{R}$$

and every k-form, ω, on U can be written uniquely as a sum

$$\omega = \sum f_I \, dx_I \tag{2.2.7}$$

where dx_I is the k-form, $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and f_I is a real-valued function,

$$f_I : U \to \mathbb{R} \,.$$

Definition 2.2.2. *The k-form (2.2.7) is of class C^r if each of the f_I's is in $C^r(U)$.*

Henceforth we'll assume, unless otherwise stated, that *all the k-forms we consider are of class $\mathcal{C}^\infty$*, and we'll denote the space of these k-forms by $\Omega^k(U)$.

We will conclude this section by discussing a few simple operations on k-forms.

1. Given a function, $f \in \mathcal{C}^\infty(U)$ and a k-form $\omega \in \Omega^k(U)$ we define $f\omega \in \Omega^k(U)$ to be the k-form

$$p \in U \to f(p)\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)\,.$$

2. Given $\omega_i \in \Omega^k(U)$, $i = 1, 2$ we define $\omega_1 + \omega_2 \in \Omega^k(U)$ to be the k-form

$$p \in U \to (\omega_1)_p + (\omega_2)_p \in \Lambda^k(T_p^*\mathbb{R}^n)\,.$$

(Notice that this sum makes sense since each summand is in $\Lambda^k(T_p^*\mathbb{R}^n)$.)

3. Given $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$ we define their *wedge product*, $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(u)$ to be the $(k_1 + k_2)$-form

$$p \in U \to (\omega_1)_p \wedge (\omega_2)_p \in \Lambda^{k_1+k_2}(T_p^*\mathbb{R}^n)\,.$$

We recall that $\Lambda^0(T_p^*\mathbb{R}^n) = \mathbb{R}$, so a zero-form is an $\mathbb{R}$-valued function and a zero form of class $\mathcal{C}^\infty$ is a $\mathcal{C}^\infty$ function, i.e.,

$$\Omega^0(U) = \mathcal{C}^\infty(U)\,.$$

A fundamental operation on forms is the "d-operation" which associates to a function $f \in \mathcal{C}^\infty(U)$ the 1-form df. It's clear from the identity (2.1.10) that df is a 1-form of class $\mathcal{C}^\infty$, so the d-operation can be viewed as a map

$$d : \Omega^0(U) \to \Omega^1(U)\,. \tag{2.2.8}$$

We will show in the next section that an analogue of this map exists for every $\Omega^k(U)$.

Exercises.

1. Let $\omega \in \Omega^2(\mathbb{R}^4)$ be the 2-form, $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Compute $\omega \wedge \omega$.

2. Let $\omega_i \in \Omega^1(\mathbb{R}^3)$, $i = 1, 2, 3$ be the 1-forms

$$\begin{aligned}\omega_1 &= x_2\,dx_3 - x_3\,dx_2\\ \omega_2 &= x_3\,dx_1 - x_1\,dx_3\end{aligned}$$

and

$$\omega_3 = x_1\,dx_2 - x_2\,dx_1\,.$$

Compute

(a) $\omega_1 \wedge \omega_2$.

(b) $\omega_2 \wedge \omega_3$.

(c) $\omega_3 \wedge \omega_1$.

(d) $\omega_1 \wedge \omega_2 \wedge \omega_3$.

3. Let U be an open subset of $\mathbb{R}^n$ and $f_i \in \mathcal{C}^\infty(U)$, $i = 1, \ldots, n$. Show that

$$df_1 \wedge \cdots \wedge df_n = \det \left[\frac{\partial f_i}{\partial x_j} \right] dx_1 \wedge \cdots \wedge dx_n \,.$$

4. Let U be an open subset of $\mathbb{R}^n$. Show that every $(n-1)$-form, $\omega \in \Omega^{n-1}(U)$, can be written uniquely as a sum

$$\sum_{i=1}^{n} f_i \; dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where $f_i \in \mathcal{C}^\infty(U)$ and the "cap" over dx_i means that dx_i is to be deleted from the product, $dx_1 \wedge \cdots \wedge dx_n$.

5. Let $\mu = \sum_{i=1}^{n} x_i dx_i$. Show that there exists an $(n-1)$-form, $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ with the property

$$\mu \wedge \omega = dx_1 \wedge \cdots \wedge dx_n \,.$$

6. Let J be the multi-index $(j_1, \ldots, j_k)$ and let $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$. Show that $dx_J = 0$ if $j_r = j_s$ for some $r \neq s$ and show that if the j_r's are all distinct

$$dx_J = (-1)^\sigma \, dx_I$$

where $I = (i_1, \ldots, i_k)$ is the strictly increasing rearrangement of $(j_1, \ldots, j_k)$ and σ is the permutation

$$j_1 \to i_1, \ldots, j_k \to i_k \,.$$

7. Let I be a strictly increasing multi-index of length k and J a strictly increasing multi-index of length ℓ. What can one say about the wedge product $dx_I \wedge dx_J$?

2.3 Exterior differentiation

Let U be an open subset of $\mathbb{R}^n$. In this section we are going to define an operation

(2.3.1) $$d : \Omega^k(U) \to \Omega^{k+1}(U) \,.$$

This operation is called *exterior differentiation* and is the fundamental operation in n-dimensional vector calculus.

For $k = 0$ we already defined the operation (2.3.1) in §2.1. Before defining it for the higher k's we list some properties that we will require to this operation to satisfy.

Property I. For ω_1 and ω_2 in $\Omega^k(U)$, $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

Property II. For $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$

(2.3.2) $$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 \,.$$

Property III. For $\omega \in \Omega^k(U)$

(2.3.3) $$d(d\omega) = 0 \,.$$

Let's point out a few consequences of these properties. First note that by Property III

(2.3.4) $$d(df) = 0$$

for every function, $f \in \mathcal{C}^\infty(U)$. More generally, given k functions, $f_i \in \mathcal{C}^\infty(U)$, $i = 1, \dots, k$, then by combining (2.3.4) with (2.3.2) we get by induction on k:

(2.3.5) $$d(df_1 \wedge \cdots \wedge df_k) = 0 \,.$$

Proof. Let $\mu = df_2 \wedge \cdots \wedge df_k$. Then by induction on k, $d\mu = 0$; and hence by (2.3.2) and (2.3.4)

$$d(df_1 \wedge \mu) = d(d_1 f) \wedge \mu + (-1)\, df_1 \wedge d\mu = 0 \,,$$

as claimed.)

In particular, given a multi-index, $I = (i_1, \dots, i_k)$ with $1 \le i_r \le n$

$$d(dx_I) = d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = 0\,. \tag{2.3.6}$$

Recall now that every k-form, $\omega \in \Omega^k(U)$, can be written uniquely as a sum

$$\omega = \sum f_I\, dx_I\,, \quad f_I \in \mathcal{C}^\infty(U)$$

where the multi-indices, I, are strictly increasing. Thus by (2.3.2) and (2.3.6)

$$d\omega = \sum df_I \wedge dx_I\,. \tag{2.3.7}$$

This shows that if there exists a "d" with properties I—III, it has to be given by the formula (2.3.7). Hence all we have to show is that the operator defined by this formula has these properties. Property I is obvious. To verify Property II we first note that for I strictly increasing (2.3.6) is a special case of (2.3.7). (Take $f_I = 1$ and $f_J = 0$ for $J \neq I$.) Moreover, if I is not strictly increasing it is either repeating, in which case $dx_I = 0$, or non-repeating in which case I^σ is strictly increasing for some permutation, $\sigma \in S_k$, and

$$dx_I = (-1)^\sigma\, dx_{I^\sigma}\,. \tag{2.3.8}$$

Hence (2.3.7) implies (2.3.6) for *all* multi-indices I. The same argument shows that for *any* sum over indices, I, for length k

$$\sum f_I dx_I$$

one has the identity:

$$d(\sum f_I\, dx_I) = \sum df_I \wedge dx_I\,. \tag{2.3.9}$$

(As above we can ignore the repeating I's, since for these I's, $dx_I = 0$, and by (2.3.8) we can make the non-repeating I's strictly increasing.)

Suppose now that $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$. Writing

$$\omega_1 = \sum f_I\, dx_I$$

and

$$\omega_2 = \sum g_J\, dx_J$$

with f_I and g_J in $\mathcal{C}^\infty(U)$ we get for the wedge product

$$\omega_1 \wedge \omega_2 = \sum f_I g_J \, dx_I \wedge dx_J \tag{2.3.10}$$

and by (2.3.9)

$$d(\omega_1 \wedge \omega_2) = \sum d(f_I g_J) \wedge dx_I \wedge dx_J \, . \tag{2.3.11}$$

(Notice that if $I = (i_1, \cdots, i_k)$ and $J = (j_i, \ldots, i_\ell)$, $dx_I \wedge dx_J = dx_K$, K being the multi-index, $(i_1, \ldots, i_k, j_1, \ldots, j_\ell)$. Even if I and J are strictly increasing, K won't necessarily be strictly increasing. However in deducing (2.3.11) from (2.3.10) we've observed that this doesn't matter .) Now note that by (2.1.11)

$$d(f_I g_J) = g_J \, df_I + f_I \, dg_J \, ,$$

and by the wedge product identities of §(1.6),

$$\begin{aligned} dg_J \wedge dx_I &= dg_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (-1)^k \, dx_I \wedge dg_J \, , \end{aligned}$$

so the sum (2.3.11) can be rewritten:

$$\sum df_I \wedge dx_I \wedge g_J \, dx_J + (-1)^k \sum f_I \, dx_I \wedge \, dg_J \wedge \, dx_J \, ,$$

or

$$\left(\sum df_I \wedge \, dx_I\right) \wedge \left(\sum g_J \, dx_J\right) + (-1)^k \left(\sum dg_J \wedge dx_J\right) \, ,$$

or finally:

$$d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 \, .$$

Thus the "d" defined by (2.3.7) has Property II. Let's now check that it has Property III. If $\omega = \sum f_I \, dx_I$, $f_I \in \mathcal{C}^\infty(U)$, then by definition, $d\omega = \sum df_I \wedge dx_I$ and by (2.3.6) and (2.3.2)

$$d(d\omega) = \sum d(df_I) \wedge \, dx_I \, ,$$

so it suffices to check that $d(df_I) = 0$, i.e., it suffices to check (2.3.4) for zero forms, $f \in \mathcal{C}^\infty(U)$. However, by (2.1.9)

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \, dx_j$$

so by (2.3.7)

$$\begin{aligned} d(df) &= \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial x_j}\right) dx_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i\right) \wedge dx_j \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \,. \end{aligned}$$

Notice, however, that in this sum, $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

so the (i,j) term cancels the (j,i) term, and the total sum is zero. □

A form, $\omega \in \Omega^k(U)$, is said to be *closed* if $d\omega = 0$ and is said to be *exact* if $\omega = d\mu$ for some $\mu \in \Omega^{k-1}(U)$. By Property III every exact form is closed, but the converse is not true even for 1-forms. (See §2.1, exercise 8). In fact it's a very interesting (and hard) question to determine if an open set, U, has the property: "For $k > 0$ every closed k-form is exact."[1]

Some examples of sets with this property are described in the exercises at the end of §2.5. We will also sketch below a proof of the following result (and ask you to fill in the details).

Lemma 2.3.1 (Poincaré's Lemma.). *If ω is a closed form on U of degree $k > 0$, then for every point, $p \in U$, there exists a neighborhood of p on which ω is exact.*

(See exercises 5 and 6 below.)

Exercises:

1. Compute the exterior derivatives of the forms below.

[1]For $k = 0$, $df = 0$ *doesn't* imply that f is exact. In fact "exactness" doesn't make much sense for zero forms since there aren't any "-1" forms. However, if $f \in \mathcal{C}^\infty(U)$ and $df = 0$ then f is constant on connected components of U. (See § 2.1, exercise 2.)

(a) $x_1\,dx_2 \wedge\, dx_3$

(b) $x_1\,dx_2 - x_2\,dx_1$

(c) $e^{-f}\,df$ where $f = \sum_{i=1}^n x_i^2$

(d) $\sum_{i=1}^n x_i\,dx_i$

(e) $\sum_{i=1}^n (-1)^i x_i\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge\, dx_n$

2. Solve the equation: $d\mu = \omega$ for $\mu \in \Omega^1(\mathbb{R}^3)$, where ω is the 2-form

(a) $dx_2 \wedge\, dx_3$

(b) $x_2\,dx_2 \wedge\, dx_3$

(c) $(x_1^2 + x_2^2)\,dx_1 \wedge\, dx_2$

(d) $\cos x_1\,dx_1 \wedge\, dx_3$

3. Let U be an open subset of $\mathbb{R}^n$.

(a) Show that if $\mu \in \Omega^k(U)$ is exact and $\omega \in \Omega^\ell(U)$ is closed then $\mu \wedge \omega$ is exact. *Hint:* The formula (2.3.2).

(b) In particular, dx_1 is exact, so if $\omega \in \Omega^\ell(U)$ is closed $dx_1 \wedge \omega = d\mu$. What is μ?

4. Let Q be the rectangle, $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Show that if ω is in $\Omega^n(Q)$, then ω is exact.

Hint: Let $\omega = f\,dx_1 \wedge \cdots \wedge\, dx_n$ with $f \in \mathcal{C}^\infty(Q)$ and let g be the function

$$g(x_1, \dots, x_n) = \int_{a_1}^{x_1} f(t, x_2, \dots, x_n)\,dt\,.$$

Show that $\omega = d(g\,dx_2 \wedge \cdots \wedge\, dx_n)$.

5. Let U be an open subset of $\mathbb{R}^{n-1}$, $A \subseteq \mathbb{R}$ an open interval and (x, t) product coordinates on $U \times A$. We will say that a form, $\mu \in \Omega^\ell(U \times A)$ is *reduced* if it can be written as a sum

$$\mu = \sum f_I(x, t)\,dx_I\,, \tag{2.3.12}$$

(i.e., no terms involving dt).

(a) Show that every form, $\omega \in \Omega^k(U \times A)$ can be written uniquely as a sum:

$$\omega = dt \wedge \alpha + \beta \tag{2.3.13}$$

where α and β are reduced.

(b) Let μ be the reduced form (2.3.12) and let

$$\frac{d\mu}{dt} = \sum \frac{d}{dt} f_I(x,t)\, dx_I$$

and

$$d_U \mu = \sum_I \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} f_I(x,t)\, dx_i \right) \wedge dx_I\,.$$

Show that

$$d\mu = dt \wedge \frac{d\mu}{dt} + d_U \mu\,.$$

(c) Let ω be the form (2.3.13). Show that

$$d\omega = dt \wedge d_U \alpha + dt \wedge \frac{d\beta}{dt} + d_U \beta$$

and conclude that ω is closed if and only if

$$\begin{aligned} \frac{d\beta}{dt} &= d_U \alpha \\ d\beta_U &= 0\,. \end{aligned} \tag{2.3.14}$$

(d) Let α be a reduced $(k-1)$-form. Show that there exists a reduced $(k-1)$-form, ν, such that

$$\frac{d\nu}{dt} = \alpha\,. \tag{2.3.15}$$

Hint: Let $\alpha = \sum f_I(x,t)\, dx_I$ and $\nu = \sum g_I(x,t)\, dx_I$. The equation (2.3.15) reduces to the system of equations

$$\frac{d}{dt} g_I(x,t) = f_I(x,t)\,. \tag{2.3.16}$$

Let c be a point on the interval, A, and using freshman calculus show that (2.3.16) has a unique solution, $g_I(x,t)$, with $g_I(x,c) = 0$.

(e) Show that if ω is the form (2.3.13) and ν a solution of (2.3.15) then the form

$$\omega - d\nu \tag{2.3.17}$$

is reduced.

(f) Let

$$\gamma = \sum h_I(x,t)\, dx)I$$

be a reduced k-form. Deduce from (2.3.14) that if γ is closed then $\frac{d\gamma}{dt} = 0$ and $d_U\gamma = 0$. Conclude that $h_I(x,t) = h_I(x)$ and that

$$\gamma = \sum h_I(x)\, dx_I$$

is effectively a closed k-form on U. Now prove: If every closed k-form on U is exact, then every closed k-form on $U \times A$ is exact. *Hint:* Let ω be a closed k-form on $U \times A$ and let γ be the form (2.3.17).

6. Let $Q \subseteq \mathbb{R}^n$ be an open rectangle. Show that every closed form on Q of degree $k > 0$ is exact. *Hint:* Let $Q = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Prove this assertion by induction, at the n^{th} stage of the induction letting $U = (a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1})$ and $A = (a_n, b_n)$.

2.4 The interior product operation

In §2.1 we explained how to pair a one-form, ω, and a vector field, v, to get a function, $\iota(v)\omega$. This pairing operation generalizes: If one is given a k-form, ω, and a vector field, v, both defined on an open subset, U, one can define a $(k-1)$-form on U by defining its value at $p \in U$ to be the interior product

$$\iota(v(p))\omega(p)\,. \tag{2.4.1}$$

Note that $v(p)$ is in $T_p\mathbb{R}^n$ and $\omega(p)$ in $\Lambda^k(T_p^*\mathbb{R}^n)$, so by definition of interior product (see §1.7), the expression (2.4.1) is an element of $\Lambda^{k-1}(T_p^*\mathbb{R}^n)$. We will denote by $\iota(v)\omega$ the $(k-1)-$form on U whose value at p is (2.4.1). From the properties of interior product on vector spaces which we discussed in §1.7, one gets analogous properties for this interior product on forms. We will list these properties, leaving their verification as an exercise. Let v and ω be vector fields, and ω_1

and ω_2 k-forms, ω a k-form and μ an ℓ-form. Then $\iota(v)\omega$ is linear in ω:

$$\iota(v)(\omega_1 + \omega_2) = \iota(v)\omega_1 + \iota(v)\omega_2\,, \tag{2.4.2}$$

linear in v:

$$\iota(v + w)\omega = \iota(v)\omega + z(w)\omega\,, \tag{2.4.3}$$

has the derivation property:

$$\iota(v)(\omega \wedge \mu) = \iota(v)\omega \wedge \mu + (-1)^k \omega \wedge \iota(v)\mu \tag{2.4.4}$$

satisfies the identity

$$\iota(v)(\iota(w)\omega) = -\iota(w)(\iota(v)\omega) \tag{2.4.5}$$

and, as a special case of (2.4.5), the identity,

$$\iota(v)(\iota(v)\omega) = 0\,. \tag{2.4.6}$$

Moreover, if ω is "decomposable" i.e., is a wedge product of one-forms

$$\omega \quad = \quad \mu_1 \wedge \cdots \wedge \mu_k\,, \tag{2.4.7}$$

then

$$\iota(v)\omega \quad = \quad \sum_{r=1}^{k} (-1)^{r-1} (\iota(v)\mu_r)\mu_1 \wedge \cdots \widehat{\mu}_r \cdots \wedge \mu_k\,. \tag{2.4.8}$$

We will also leave for you to prove the following two assertions, both of which are special cases of (2.4.8). If $v = \partial/\partial x_r$ and $\omega = dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ then

$$\iota(v)\omega \quad = \quad \sum_{r=1}^{k} (-1)^r \delta^i_{i_r}\, dx_{I_r} \tag{2.4.9}$$

where

$$\delta^i_{i_r} \quad = \quad \begin{cases} 1 & i = i_r \\ 0\,, & i \neq i_r \end{cases}\,.$$

and $I_r = (i_1, \dots, \widehat{i}_r, \dots, i_k)$ and if $v = \sum f_i\, \partial/\partial x_i$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$ then

$$\iota(v)\omega = \sum (-1)^{r-1} f_r\, dx_1 \wedge \cdots \widehat{dx}_r \cdots \wedge dx_n\,. \tag{2.4.10}$$

By combining exterior differentiation with the interior product operation one gets another basic operation of vector fields on forms: the *Lie differentiation* operation. For zero-forms, i.e., for $\mathcal{C}^\infty$ functions, φ, we defined this operation by the formula (2.1.14). For k-forms we'll define it by the slightly more complicated formula

$$L_v\omega = \iota(v)\, d\omega + \, d\iota(v)\omega\,. \tag{2.4.11}$$

(Notice that for zero-forms the second summand is zero, so (2.4.11) and (2.1.14) agree.) If ω is a k-form the right hand side of (2.4.11) is as well, so L_v takes k-forms to k-forms. It also has the property

$$dL_v\omega = L_v\, d\omega \tag{2.4.12}$$

i.e., it "commutes" with d, and the property

$$L_v(\omega \wedge \mu) = L_v\omega \wedge \mu + \omega \wedge L_v\mu \tag{2.4.13}$$

and from these properties it is fairly easy to get an explicit formula for $L_v\omega$. Namely let ω be the k-form

$$\omega = \sum f_I\, dx_I\,, \quad f_I \in \mathcal{C}^\infty(U)$$

and v the vector field

$$\sum g_i\, \partial/\partial x_i\,, \quad g_i \in \mathcal{C}^\infty(U)\,.$$

By (2.4.13)

$$L_v(f_I\, dx_I) \quad = \quad (L_v f_I)\, dx_I + f_I(L_v\, dx_I)$$

and

$$L_v\, dx_I \quad = \quad \sum_{r=1}^{k} dx_{i_1} \wedge \cdots \wedge L_v\, dx_{i_r} \wedge \cdots \wedge dx_{i_k}\,,$$

and by (2.4.12)

$$L_v\, dx_{i_r} = dL_v x_{i_r}$$

so to compute $L_v\omega$ one is reduced to computing $L_v x_{i_r}$ and $L_v f_I$. However by (2.4.13)

$$L_v x_{i_r} \;=\; g_{i_r}$$

and

$$L_v f_I \;=\; \sum g_i \frac{\partial f_I}{\partial x_i}\,.$$

We will leave the verification of (2.4.12) and (2.4.13) as exercises, and also ask you to prove (by the method of computation that we've just sketched) the *divergence formula*

$$L_v(dx_1 \wedge \cdots \wedge dx_n) = \sum \left(\frac{\partial g_i}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_n\,. \tag{2.4.14}$$

Exercises:

1. Verify the assertions (2.4.2)—(2.4.7).

2. Show that if ω is the k-form, dx_I and v the vector field, $\partial/\partial x_r$, then $\iota(v)\omega$ is given by (2.4.9).

3. Show that if ω is the n-form, $dx_1 \wedge \cdots \wedge dx_n$, and v the vector field, $\sum f_i\,\partial/\partial x_i$, $\iota(v)\omega$ is given by (2.4.10).

4. Let U be an open subset of $\mathbb{R}^n$ and v a $\mathcal{C}^\infty$ vector field on U. Show that for $\omega \in \Omega^k(U)$

$$dL_v\omega \;=\; L_v\,d\omega$$

and

$$\iota_v L_v\omega \;=\; L_v\iota_v\omega\,.$$

Hint: Deduce the first of these identities from the identity $d(d\omega) = 0$ and the second from the identity $\iota(v)(\iota(v)\omega) = 0\,.$)

5. Given $\omega_i \in \Omega^{k_i}(U)$, $i = 1, 2$, show that

$$L_v(\omega_1 \wedge \omega_2) = L_v\omega_1 \wedge \omega_2 + \omega_1 \wedge L_v\omega_2\,.$$

Hint: Plug $\omega = \omega_1 \wedge \omega_2$ into (2.4.11) and use (2.3.2) and (2.4.4)to evaluate the resulting expression.

6. Let v_1 and v_2 be vector fields on U and let w be their Lie bracket. Show that for $\omega \in \Omega^k(U)$

$$L_w\omega = L_{v_1}(L_{v_2}\omega) - L_{v_2}(L_{v_1}\omega)\,.$$

Hint: By definition this is true for zero-forms and by (2.4.12) for exact one-forms. Now use the fact that every form is a sum of wedge products of zero-forms and one-forms and the fact that L_v satisfies the product identity (2.4.13).

7. Prove the divergence formula (2.4.14).

8. (a) Let $\omega = \Omega^k(\mathbb{R}^n)$ be the form

$$\omega = \sum f_I(x_1, \dots, x_n)\, dx_I$$

and $\mathfrak{v}$ the vector field, $\partial/\partial x_n$. Show that

$$L_{\mathfrak{v}}\omega = \sum \frac{\partial}{\partial x_n} f_I(x_1, \dots, x_n)\, dx_I\,.$$

(b) Suppose $\iota(\mathfrak{v})\omega = L_{\mathfrak{v}}\omega = 0$. Show that ω only depends on $x_1, \dots, x_{k-1}$ and $dx_1, \dots, dx_{k-1}$, i.e., is effectively a k-form on $\mathbb{R}^{n-1}$.

(c) Suppose $\iota(\mathfrak{v})\omega = d\omega = 0$. Show that ω is effectively a closed k-form on $\mathbb{R}^{n-1}$.

(d) Use these results to give another proof of the Poincaré lemma for $\mathbb{R}^n$. Prove by induction on n that every closed form on $\mathbb{R}^n$ is exact.

Hints:

i. Let ω be the form in part (a) and let

$$g_I(x_1, \dots, x_n) = \int_0^{x_n} f_I(x_1, \dots, x_{n-1}, t)\, dt\,.$$

Show that if $\nu = \sum g_I\, dx_I$, then $L_{\mathfrak{v}}\nu = \omega$.

ii. Conclude that

$$(*) \qquad \omega - d\iota(\mathfrak{v})\nu = \iota(\mathfrak{v})\, d\nu\,.$$

iii. Suppose $d\omega = 0$. Conclude from (*) and from the formula (2.4.6) that the form $\beta = \iota(\mathfrak{v})\, d\nu$ satisfies $d\beta = \iota(\mathfrak{v})\beta = 0$.

iv. By part c, β is effectively a closed form on $\mathbb{R}^{n-1}$, and by induction, $\beta = d\alpha$. Thus by (*)

$$\omega = d\iota(\mathfrak{v})\nu + d\alpha\,.$$

2.5 The pull-back operation on forms

Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$ and $f : U \to V$ a $\mathcal{C}^\infty$ map. Then for $p \in U$ and $q = f(p)$, the derivative of f at p

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

is a linear map, so (as explained in §7 of Chapter 1) one gets from it a pull-back map

$$df_p^* : \Lambda^k(T_q^*\mathbb{R}^m) \to \Lambda^k(T_p^*\mathbb{R}^n)\,. \tag{2.5.1}$$

In particular, let ω be a k-form on V. Then at $q \in V$, ω takes the value

$$\omega_q \in \Lambda^k(T_q^*\mathbb{R}^m)\,,$$

so we can apply to it the operation (2.5.1), and this gives us an element:

$$df_p^*\omega_q \in \Lambda^k(T_p^*\mathbb{R}^n)\,. \tag{2.5.2}$$

In fact we can do this for every point $p \in U$, so this gives us a function,

$$p \in U \to (df_p)^*\omega_q\,, \quad q = f(p)\,. \tag{2.5.3}$$

By the definition of k-form such a function is a k-form on U. We will denote this k-form by $f^*\omega$ and define it to be the *pull-back of* ω *by the map* f. A few of its basic properties are described below.

1. Let φ be a zero-form, i.e., a function, $\varphi \in \mathcal{C}^\infty(V)$. Since

$$\Lambda^0(T_p^*) = \Lambda^0(T_q^*) = \mathbb{R}$$

the map (2.5.1) is just the identity map of $\mathbb{R}$ onto $\mathbb{R}$ when k is equal to zero. Hence for zero-forms

$$(f^*\varphi)(p) = \varphi(q)\,, \tag{2.5.4}$$

i.e., $f^*\varphi$ is just the composite function, $\varphi \circ f \in \mathcal{C}^\infty(U)$.

2. Let $\mu \in \Omega^1(V)$ be the 1-form, $\mu = d\varphi$. By the chain rule (2.5.2) unwinds to:

$$(df_p)^* d\varphi_q = (d\varphi)_q \circ df_p = d(\varphi \circ f)_p \tag{2.5.5}$$

and hence by (2.5.4)

$$f^*\, d\varphi = df^*\varphi\,. \tag{2.5.6}$$

3. If ω_1 and ω_2 are in $\Omega^k(V)$ we get from (2.5.2)

$$(df_p)^*(\omega_1+\omega_2)_q = (df_p)^*(\omega_1)_q + (df_p)^*(\omega_2)_q\,,$$

and hence by (2.5.3)

$$f^*(\omega_1+\omega_2) = f^*\omega_1 + f^*\omega_2\,.$$

4. We observed in § 1.7 that the operation (2.5.1) commutes with wedge-product, hence if ω_1 is in $\Omega^k(V)$ and ω_2 is in $\Omega^\ell(V)$

$$df_p^*(\omega_1)_q \wedge (\omega_2)_q = df_p^*(\omega_1)_q \wedge df_p^*(\omega_2)_q\,.$$

In other words

$$f^*\omega_1 \wedge \omega_2 = f^*\omega_1 \wedge f^*\omega_2\,. \tag{2.5.7}$$

5. Let W be an open subset of $\mathbb{R}^k$ and $g : V \to W$ a $\mathcal{C}^\infty$ map. Given a point $p \in U$, let $q = f(p)$ and $w = g(q)$. Then the composition of the map

$$(df_p)^* : \Lambda^k(T_q^*) \to \Lambda^k(T_p^*)$$

and the map

$$(dg_q)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_q^*)$$

is the map

$$(dg_q \circ df_p)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)$$

by formula (1.7.4) of Chapter 1. However, by the chain rule

$$(dg_q) \circ (df)_p = d(g \circ f)_p$$

so this composition is the map

$$d(g\circ f)_p^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)\,.$$

Thus if ω is in $\Omega^k(W)$

$$f^*(g^*\omega) = (g\circ f)^*\omega\,. \tag{2.5.8}$$

Let's see what the pull-back operation looks like in coordinates. Using multi-index notation we can express every k-form, $\omega \in \Omega^k(V)$ as a sum over multi-indices of length k

$$\omega = \sum \varphi_I\, dx_I\,, \tag{2.5.9}$$

the coefficient, φ_I, of dx_I being in $\mathcal{C}^\infty(V)$. Hence by (2.5.4)

$$f^*\omega = \sum f^*\varphi_I f^*(dx_I)$$

where $f^*\varphi_I$ is the function of $\varphi \circ f$. What about $f^*\, dx_I$? If I is the multi-index, $(i_1, \ldots, i_k)$, then by definition

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

so

$$d^*\, dx_I = f^*\, dx_i \wedge \cdots \wedge f^*\, dx_{i_k}$$

by (2.5.7), and by (2.5.6)

$$f^*\, dx_i = df^* x_i = df_i$$

where f_i is the i^{th} coordinate function of the map f. Thus, setting

$$df_I = df_{i_1} \wedge \cdots \wedge df_{i_k}\,,$$

we get for each multi-index, I,

$$f^*\, dx_I = df_I \tag{2.5.10}$$

and for the pull-back of the form (2.5.9)

$$f^*\omega = \sum f^*\varphi_I\, df_I\,. \tag{2.5.11}$$

We will use this formula to prove that pull-back commutes with exterior differentiation:

$$d\, f^*\omega = f^*\, d\omega\,. \tag{2.5.12}$$

To prove this we recall that by (2.2.5), $d(\,df_I) = 0$, hence by (2.2.2) and (2.5.10)

$$\begin{aligned} d\, f^*\omega &= \sum d\, f^*\varphi_I \wedge df_I \\ &= \sum f^*\, d\varphi_I \wedge df^*\, dx_I \\ &= f^* \sum d\varphi_I \wedge dx_I \\ &= f^*\, d\omega\,. \end{aligned}$$

A special case of formula (2.5.10) will be needed in Chapter 4: Let U and V be open subsets of $\mathbb{R}^n$ and let $\omega = dx_1 \wedge \cdots \wedge dx_n$. Then by (2.5.10)

$$f^*\omega_p = (df_1)_p \wedge \cdots \wedge (df_n)_p$$

for all $p \in U$. However,

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(p)(dx_j)_p$$

and hence by formula (1.7.7) of Chapter 1

$$f^*\omega_p = \det\left[\frac{\partial f_i}{\partial x_j}(p)\right](dx_1 \wedge \cdots \wedge dx_n)_p\,.$$

In other words

$$f^*\, dx_1 \wedge \cdots \wedge dx_n = \det\left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n\,. \tag{2.5.13}$$

We will outline in exercises 4 and 5 below the proof of an important topological property of the pull-back operation. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $A \subseteq \mathbb{R}$ an open interval containing 0 and 1 and $f_i : U \to V$, $i = 0, 1$, a $\mathcal{C}^\infty$ map.

Definition 2.5.1. *A $\mathcal{C}^\infty$ map, $F : U \times A \to V$, is a homotopy between f_0 and f_1 if $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.*

Thus, intuitively, f_0 and f_1 are *homotopic* if there exists a family of $\mathcal{C}^\infty$ maps, $f_t : U \to V$, $f_t(x) = F(x, t)$, which “smoothly deform f_0 into f_1”. In the exercises mentioned above you will be asked to verify that for f_0 and f_1 to be homotopic they have to satisfy the following criteria.

Theorem 2.5.2. *If f_0 and f_1 are homotopic then for every closed form, $\omega \in \Omega^k(V)$, $f_1^*\omega - f_0^*\omega$ is exact.*

This theorem is closely related to the Poincaré lemma, and, in fact, one gets from it a slightly stronger version of the Poincaré lemma than that described in exercises 5–6 in §2.2.

Definition 2.5.3. *An open subset, U, of $\mathbb{R}^n$ is* contractable *if, for some point $p_0 \in U$, the identity map*

$$f_1 : U \to U\,, \quad f(p) = p\,,$$

is homotopic to the constant map

$$f_0 : U \to U\,, \quad f_0(p) = p_0\,.$$

From the theorem above it's easy to see that the Poincaré lemma holds for contractable open subsets of $\mathbb{R}^n$. If U is contractable every closed k-form on U of degree $k > 0$ is exact. (Proof: Let ω be such a form. Then for the identity map $f_0^*\omega = \omega$ and for the constant map, $f_0^*\omega = 0$.)

Exercises.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2, x_3) = (x_1x_2, x_2x_3^2, x_3^3)\,.$$

Compute the pull-back, $f^*\omega$ for

(a) $\omega = x_2\,dx_3$

(b) $\omega = x_1\,dx_1 \wedge dx_3$

(c) $\omega = x_1\,dx_1 \wedge dx_2 \wedge dx_3$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2)\,.$$

Complete the pull-back, $f^*\omega$, for

(a) $\omega = x_2\,dx_2 + x_3\,dx_3$

(b) $\omega = x_1\,dx_2 \wedge dx_3$

(c) $\omega = dx_1 \wedge dx_2 \wedge dx_3$

3. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $f : U \to V$ a $\mathcal{C}^\infty$ map and $\gamma : [a,b] \to U$ a $\mathcal{C}^\infty$ curve. Show that for $\omega \in \Omega^1(V)$

$$\int_\gamma f^*\omega = \int_{\gamma_1} \omega$$

where $\gamma_1 : [a,b] \to V$ is the curve, $\gamma_1(t) = f(\gamma(t))$. (See § 2.1, exercise 7.)

4. Let U be an open subset of $\mathbb{R}^n$, $A \subseteq \mathbb{R}$ an open interval containing the points, 0 and 1, and (x,t) product coordinates on $U \times A$. Recall (§ 2.2, exercise 5) that a form, $\mu \in \Omega^\ell(U \times A)$ is *reduced* if it can be written as a sum

$$\mu = \sum f_I(x,t)\, dx_I \tag{2.5.14}$$

(i.e., none of the summands involve "dt"). For a reduced form, μ, let $Q\mu \in \Omega^\ell(U)$ be the form

$$Q\mu = \left(\sum \int_0^1 f_I(x,t)\, dt\right) dx_I \tag{2.5.15}$$

and let $\mu_i \in \Omega^\ell(U)$, $i = 0,1$ be the forms

$$\mu_0 \;=\; \sum f_I(x,0)\, dx_I \tag{2.5.16}$$

and

$$\mu_1 \;=\; \sum f_I(x,1)\, dx_I\,. \tag{2.5.17}$$

Now recall that every form, $\omega \in \Omega^k(U \times A)$ can be written uniquely as a sum

$$\omega = \; dt \wedge \alpha + \beta \tag{2.5.18}$$

where α and β are reduced. (See exercise 5 of § 2.3, part a.)

(a) Prove

Theorem 2.5.4. *If the form (2.5.18) is closed then*

$$\beta_0 - \beta_1 = dQ\alpha\,. \tag{2.5.19}$$

Hint: *Formula (2.3.14).*

(b) Let ι_0 and ι_1 be the maps of U into $U \times A$ defined by $\iota_0(x) = (x,0)$ and $\iota_1(x) = (x,1)$. Show that (2.5.19) can be rewritten

$$\iota_0^*\omega - \iota_1^*\omega = \; dQ\alpha\,. \tag{2.5.20}$$

5. Let V be an open subset of $\mathbb{R}^m$ and $f_i : U \to V$, $i = 0,1$, $\mathcal{C}^\infty$ maps. Suppose f_0 and f_1 are homotopic. Show that for every closed form, $\mu \in \Omega^k(V)$, $f_1^*\mu - f_0^*\mu$ is exact. *Hint:* Let $F : U \times A \to V$ be a

homotopy between f_0 and f_1 and let $\omega = F^*\mu$. Show that ω is closed and that $f_0^*\mu = \iota_0^*\omega$ and $f_1^*\mu = \iota_1^*\omega$. Conclude from (2.5.20) that

$$f_0^*\mu - f_1^*\mu = dQ\alpha \tag{2.5.21}$$

where $\omega = dt \wedge \alpha + \beta$ and α and β are reduced.

6. Show that if $U \subseteq \mathbb{R}^n$ is a contractable open set, then the Poincaré lemma holds: every closed form of degree $k > 0$ is exact.

7. An open subset, U, of $\mathbb{R}^n$ is said to be *star-shaped* if there exists a point $p_0 \in U$, with the property that for every point $p \in U$, the line segment,

$$tp + (1-t)p_0\,, \quad 0 \le t \le 1\,,$$

joining p to p_0 is contained in U. Show that if U is star-shaped it is contractable.

8. Show that the following open sets are star-shaped:

(a) The open unit ball

$$\{x \in \mathbb{R}^n\,,\ \|x\| < 1\}\,.$$

(b) The open rectangle, $I_1 \times \cdots \times I_n$, where each I_k is an open subinterval of $\mathbb{R}$.

(c) $\mathbb{R}^n$ itself.

(d) Product sets

$$U_1 \times U_2 \subseteq \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$$

where U_i is a star-shaped open set in $\mathbb{R}^{n_i}$.

9. Let U be an open subset of $\mathbb{R}^n$, $f_t : U \to U$, $t \in \mathbb{R}$, a one-parameter group of diffeomorphisms and v its infinitesimal generator. Given $\omega \in \Omega^k(U)$ show that at $t = 0$

$$\frac{d}{dt} f_t^*\omega = L_v\omega\,. \tag{2.5.22}$$

Here is a sketch of a proof:

(a) Let $\gamma(t)$ be the curve, $\gamma(t) = f_t(p)$, and let φ be a zero-form, i.e., an element of $\mathcal{C}^\infty(U)$. Show that

$$f_t^*\varphi(p) = \varphi(\gamma(t))$$

and by differentiating this identity at $t = 0$ conclude that (2.4.40) holds for zero-forms.

(b) Show that if (2.4.40) holds for ω it holds for $d\omega$. *Hint:* Differentiate the identity

$$f_t^*\, d\omega = df_t^*\omega$$

at $t = 0$.

(c) Show that if (2.4.40) holds for ω_1 and ω_2 it holds for $\omega_1 \wedge \omega_2$. *Hint:* Differentiate the identity

$$f_t^*(\omega_1 \wedge \omega_2) = f_t^*\omega_1 \wedge f_t^*\omega_2$$

at $t = 0$.

(d) Deduce (2.4.40) from a, b and c. *Hint:* Every k-form is a sum of wedge products of zero-forms and exact one-forms.

10. In exercise 9 show that for *all* t

$$\frac{d}{dt} f_t^*\omega = f_t^* L_v\omega = L_v f_t^*\omega\,. \tag{2.5.23}$$

Hint: By the definition of "one-parameter group", $f_{s+t} = f_s \circ f_t = f_r \circ f_s$, hence:

$$f_{s+t}^*\omega = f_t^*(f_s^*\omega) = f_s^*(f_t^*\omega)\,.$$

Prove the first assertion by differentiating the first of these identities with respect to s and then setting $s = 0$, and prove the second assertion by doing the same for the second of these identities.

In particular conclude that

$$f_t^* L_v\omega = L_v f_t^*\omega\,. \tag{2.5.24}$$

11. (a) By massaging the result above show that

$$\frac{d}{dt} f_t^*\omega \;=\; dQ_t\omega + Q_t\, d\omega \tag{2.5.25}$$

where

$$Q_t\omega \;=\; f_t^*\iota(v)\omega\,. \tag{2.5.26}$$

Hint: Formula (2.4.11).

(b) Let

$$Q\omega = \int_0^1 f_t^* \iota(v)\omega \, dt \, .$$

Prove the homotopy indentity

$$f_1^*\omega - f_0^*\omega = \, dQ\omega + Q \, d\omega \, . \tag{2.5.27}$$

12. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, v a vector field on U, w a vector field on V and $f : U \to V$ a $\mathcal{C}^\infty$ map. Show that if v and w are f-related

$$\iota(v) f^* \omega = f^* \iota(w) \omega \, .$$

Hint: Chapter 1, §1.7, exercise 8.

2.6 Div, curl and grad

The basic operations in 3-dimensional vector calculus: grad, curl and div are, by definition, operations on *vector fields*. As we'll see below these operations are closely related to the operations

$$d : \Omega^k(\mathbb{R}^3) \to \Omega^{k+1}(\mathbb{R}^3) \tag{2.6.1}$$

in degrees $k = 0, 1, 2$. However, only two of these operations: grad and div, generalize to n dimensions. (They are essentially the d-operations in degrees zero and $n-1$.) And, unfortunately, there is no simple description in terms of vector fields for the other $n-2$ d-operations. This is one of the main reasons why an adequate theory of vector calculus in n-dimensions forces on one the differential form approach that we've developed in this chapter. Even in three dimensions, however, there is a good reason for replacing grad, div and curl by the three operations, (2.6.1). A problem that physicists spend a lot of time worrying about is the problem of *general covariance*: formulating the laws of physics in such a way that they admit as large a set of symmetries as possible, and frequently these formulations involve differential forms. An example is Maxwell's equations, the fundamental laws of electromagnetism. These are usually expressed as identities involving div and curl. However, as we'll explain below, there is an alternative formulation of Maxwell's equations based on the operations (2.6.1), and from the point of view of general covariance, this formulation is much more satisfactory: the only symmetries of $\mathbb{R}^3$ which preserve div and curl are translations and rotations, whereas the operations (2.6.1) admit all diffeomorphisms of $\mathbb{R}^3$ as symmetries.

To describe how grad, div and curl are related to the operations (2.6.1) we first note that there are two ways of converting vector fields into forms. The first makes use of the natural inner product, $B(v,w) = \sum v_i w_i$, on $\mathbb{R}^n$. From this inner product one gets by § 1.2, exercise 9 a bijective linear map:

$$L : \mathbb{R}^n \to (\mathbb{R}^n)^* \tag{2.6.2}$$

with the defining property: $L(v) = \ell \Leftrightarrow \ell(w) = B(v,w)$. Via the identification (2.1.2) B and L can be transferred to $T_p\mathbb{R}^n$, giving one an inner product, B_p, on $T_p\mathbb{R}^n$ and a bijective linear map

$$L_p : T_p\mathbb{R}^n \to T_p^*\mathbb{R}^n \,. \tag{2.6.3}$$

Hence if we're given a vector field, $\mathfrak{v}$, on U we can convert it into a 1-form, $\mathfrak{v}^\sharp$, by setting

$$\mathfrak{v}^\sharp(p) = L_p \mathfrak{v}(p) \tag{2.6.4}$$

and this sets up a one–one correspondence between vector fields and 1-forms. For instance

$$\mathfrak{v} = \frac{\partial}{\partial x_i} \Leftrightarrow \mathfrak{v}^\sharp = dx_i \,, \tag{2.6.5}$$

(see exercise 3 below) and, more generally,

$$\mathfrak{v} = \sum f_i \frac{\partial}{\partial x_i} \Leftrightarrow \mathfrak{v}^\sharp = \sum f_i \, dx_i \,. \tag{2.6.6}$$

In particular if f is a $\mathcal{C}^\infty$ function on U the vector field "grad f" is by definition

$$\sum \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \tag{2.6.7}$$

and this gets converted by (2.6.8) into the 1-form, df. Thus the "grad" operation in vector calculus is basically just the operation, $d : \Omega^0(U) \to \Omega^1(U)$.

The second way of converting vector fields into forms is via the interior product operation. Namely let Ω be the n-form, $dx_1 \wedge \cdots \wedge dx_n$. Given an open subset, U of $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ vector field,

$$v = \sum f_i \frac{\partial}{\partial x_i} \tag{2.6.8}$$

on U the interior product of v with Ω is the $(n-1)$-form

$$\iota(v)\Omega = \sum (-1)^{r-1} f_r dx_1 \wedge \cdots \wedge \widehat{dx}_r \cdots \wedge dx_n \,. \tag{2.6.9}$$

Moreover, every $(n-1)$-form can be written uniquely as such a sum, so (2.6.8) and (2.6.9) set up a one-one correspondence between vector fields and $(n-1)$-forms. Under this correspondence the d-operation gets converted into an operation on vector fields

$$v \to d\iota(v)\Omega \,. \tag{2.6.10}$$

Moreover, by (2.4.11)

$$d\iota(v)\Omega = L_v\Omega$$

and by (2.4.14)

$$L_v\Omega = \mathrm{div}(v)\Omega$$

where

$$\mathrm{div}(v) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \,. \tag{2.6.11}$$

In other words, this correspondence between $(n-1)$-forms and vector fields converts the d-operation into the divergence operation (2.6.11) on vector fields.

Notice that “div” and “grad” are well-defined as vector calculus operations in n-dimensions even though one usually thinks of them as operations in 3-dimensional vector calculus. The “curl” operation, however, is intrinsically a 3-dimensional vector calculus operation. To define it we note that by (2.6.9) every 2-form, μ, can be written uniquely as an interior product,

$$\mu = \iota(\mathfrak{w})\, dx_1 \wedge dx_2 \wedge dx_3 \,, \tag{2.6.12}$$

for some vector field $\mathfrak{w}$, and the left-hand side of this formula determines $\mathfrak{w}$ uniquely. Now let U be an open subset of $\mathbb{R}^3$ and $\mathfrak{v}$ a vector field on U. From $\mathfrak{v}$ we get by (2.6.6) a 1-form, $\mathfrak{v}^\sharp$, and hence by (2.6.12) a vector field, $\mathfrak{w}$, satisfying

$$d\mathfrak{v}^\sharp = \iota(\mathfrak{w})\, dx_1 \wedge dx_2 \wedge dx_3 \,. \tag{2.6.13}$$

The "curl" of $\mathfrak{v}$ is defined to be this vector field, in other words,

$$\operatorname{curl} \mathfrak{v} = \mathfrak{w} \,, \tag{2.6.14}$$

where $\mathfrak{v}$ and $\mathfrak{w}$ are related by (2.6.13).

We'll leave for you to check that this definition coincides with the definition one finds in calculus books. More explicitly we'll leave for you to check that if v is the vector field

$$v = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3} \tag{2.6.15}$$

then

$$\operatorname{curl} v = g_1 \frac{\partial}{\partial x_1} + g_2 \frac{\partial}{\partial x_2} + g_3 \frac{\partial}{\partial x_3} \tag{2.6.16}$$

where

$$\begin{aligned} g_1 &= \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} \\ g_2 &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \\ g_3 &= \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \,. \end{aligned} \tag{2.6.17}$$

To summarize: the grad, curl and div operations in 3-dimensions are basically just the three operations (2.6.1). The "grad" operation is the operation (2.6.1) in degree zero, "curl" is the operation (2.6.1) in degree one and "div" is the operation (2.6.1) in degree two. However, to define "grad" we had to assign an inner product, B_p, to the next tangent space, $T_p\mathbb{R}^n$, for each p in U; to define "div" we had to equip U with the 3-form, Ω, and to define "curl", the most complicated of these three operations, we needed the B_p's *and* Ω. This is why diffeomorphisms preserve the three operations (2.6.1) but don't preserve grad, curl and div. The additional structures which one needs to define grad, curl and div are only preserved by translations and rotations.

We will conclude this section by showing how Maxwell's equations, which are usually formulated in terms of div and curl, can be reset into "form" language. (The paragraph below is an abbreviated version of Guillemin–Sternberg, *Symplectic Techniques in Physics*, §1.20.)

Maxwell's equations assert:

$$\operatorname{div} \mathfrak{v}_E = q \tag{2.6.18}$$

$$\operatorname{curl} \mathfrak{v}_E = -\frac{\partial}{\partial t}\mathfrak{v}_M \tag{2.6.19}$$

$$\operatorname{div} \mathfrak{v}_M = 0 \tag{2.6.20}$$

$$c^2 \operatorname{curl} \mathfrak{v}_M = \mathfrak{w} + \frac{\partial}{\partial t}\mathfrak{v}_E \tag{2.6.21}$$

where $\mathfrak{v}_E$ and $\mathfrak{v}_M$ are the *electric* and *magnetic* fields, q is the *scalar charge density*, $\mathfrak{w}$ is the *current density* and c is the velocity of light. (To simplify (2.6.25) slightly we'll assume that our units of space–time are chosen so that $c = 1$.) As above let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ and let

$$\mu_E = \iota(\mathfrak{v}_E)\Omega \tag{2.6.22}$$

and

$$\mu_M = \iota(\mathfrak{v}_M)\Omega\,. \tag{2.6.23}$$

We can then rewrite equations (2.6.18) and (2.6.20) in the form

$$d\mu_E = q\Omega \tag{2.6.18$'$}$$

and

$$d\mu_M = 0\,. \tag{2.6.20$'$}$$

What about (2.6.19) and (2.6.21)? We will leave the following "form" versions of these equations as an exercise.

$$d\mathfrak{v}_E^\sharp = -\frac{\partial}{\partial t}\mu_M \tag{2.6.19$'$}$$

and

$$d\mathfrak{v}_M^\sharp = \iota(\mathfrak{w})\Omega + \frac{\partial}{\partial t}\mu_E \tag{2.6.21$'$}$$

where the 1-forms, $\mathfrak{v}_E^\sharp$ and $\mathfrak{v}_M^\sharp$, are obtained from $\mathfrak{v}_E$ and $\mathfrak{v}_M$ by the operation, (2.6.4).

These equations can be written more compactly as differential form identities in $3+1$ dimensions. Let ω_M and ω_E be the 2-forms

$$\omega_M = \mu_M - \mathfrak{v}_E^\sharp \wedge dt \tag{2.6.24}$$

and

$$\omega_E = \mu_E - \mathfrak{v}_M^\sharp \wedge dt \tag{2.6.25}$$

and let Λ be the 3-form

$$\Lambda = q\Omega + \iota(\mathfrak{w})\Omega \wedge dt\,. \tag{2.6.26}$$

We will leave for you to show that the four equations (2.6.18) — (2.6.21) are equivalent to two elegant and compact (3+1)-dimensional identities

$$d\omega_M = 0 \tag{2.6.27}$$

and

$$d\omega_E = \Lambda\,. \tag{2.6.28}$$

Exercises.

1. Verify that the "curl" operation is given in coordinates by the formula (2.6.17).

2. Verify that the Maxwell's equations, (2.6.18) and (2.6.19) become the equations (2.6.20) and (2.6.21) when rewritten in differential form notation.

3. Show that in $(3+1)$-dimensions Maxwell's equations take the form (2.6.17)–(2.6.18).

4. Let U be an open subset of $\mathbb{R}^3$ and v a vector field on U. Show that if v is the gradient of a function, its curl has to be zero.

5. If U is simply connected prove the converse: If the curl of v vanishes, v is the gradient of a function.

6. Let $w = \operatorname{curl} v$. Show that the divergence of w is zero.

7. Is the converse statment true? Suppose the divergence of w is zero. Is $w = \operatorname{curl} v$ for some vector field v?

2.7 Symplectic geometry and classical mechanics

In this section we'll describe some other applications of the theory of differential forms to physics. Before describing these applications, however, we'll say a few words about the geometric ideas that are involved. Let $x_1, \dots, x_{2n}$ be the standard coordinate functions on $\mathbb{R}^{2n}$ and for $i = 1, \dots, n$ let $y_i = x_{i+n}$. The two-form

$$\omega = \sum_{i=1}^{n} dx_i \wedge jy_i \tag{2.7.1}$$

is known as the *Darboux* form. From the identity

$$\omega = -d\left(\sum y_i \, dx_i\right) . \tag{2.7.2}$$

it follows that ω is exact. Moreover computing the n-fold wedge product of ω with itself we get

$$\begin{aligned} \omega^n &= \left(\sum_{i_i=1}^{n} dx_{i_1} \wedge dy_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^{n} dx_{i_n} \wedge dy_{i_n}\right) \\ &= \sum_{i_1,\dots,i_n} dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dy_{i_n} . \end{aligned}$$

We can simplify this sum by noting that if the multi-index, $I = i_1, \dots, i_n$, is repeating the wedge product

$$dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dx_{i_n} \tag{2.7.3}$$

involves two repeating dx_{i_1}'s and hence is zero, and if I is non-repeating we can permute the factors and rewrite (2.7.3) in the form

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n .$$

(See §1.6, exercise 5.) Hence since these are exactly $n!$ non-repeating multi-indices

$$\omega^n = n! \, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

i.e.,

$$\frac{1}{n!}\omega^n = \Omega \tag{2.7.4}$$

where

$$\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \tag{2.7.5}$$

is the *symplectic volume form* on $\mathbb{R}^{2n}$.

Let U and V be open subsets of $\mathbb{R}^{2n}$. A diffeomorphism $f : U \to V$ is said to be a *symplectic* diffeomorphism (or *symplectomorphism* for short) if $f^*\omega = \omega$. In particular let

$$f_t : U \to U\,, \quad -\infty < t < \infty \tag{2.7.6}$$

be a one-parameter group of diffeomorphisms and let v be the vector field generating (2.7.6). We will say that v is a *symplectic* vector field if the diffeomorphisms, (2.7.6) are symplectomorphisms, i.e., for all t,

$$f_t^*\omega = \omega\,. \tag{2.7.7}$$

Let's see what such vector fields have to look like. Note that by (2.5.23)

$$\frac{d}{dt}f_t^*\omega = f_t^* L_v\omega\,, \tag{2.7.8}$$

hence if $f_t^*\omega = \omega$ for all t, the left hand side of (2.7.8) is zero, so

$$f_t^* L_v\omega = 0\,.$$

In particular, for $t = 0$, f_t is the identity map so $f_t^* L_v\omega = L_v\omega = 0$. Conversely, if $L_v\omega = 0$, then $f_t^* L_v\omega = 0$ so by (2.7.8) $f_t^*\omega$ doesn't depend on t. However, since $f_t^*\omega = \omega$ for $t = 0$ we conclude that $f_t^*\omega = \omega$ for all t. Thus to summarize we've proved

Theorem 2.7.1. *Let $f_t : U \to U$ be a one-parameter group of diffeomorphisms and v the infinitesmal generator of this group. Then v is symplectic of and only if $L_v\omega = 0$.*

There is an equivalent formulation of this result in terms of the interior product, $\iota(v)\omega$. By (2.4.11)

$$L_v\omega = d\iota(v)\omega + \iota(v)\,d\omega\,.$$

But by (2.7.2) $d\omega = 0$ so

$$L_v\omega = d\iota(v)\omega\,.$$

Thus we've shown

Theorem 2.7.2. *The vector field v is symplectic if and only if $\iota(v)\omega$ is closed.*

If $\iota(v)\omega$ is not only closed but is exact we'll say that v is a *Hamiltonian* vector field. In other words v is Hamiltonian if

$$\iota(v)\omega = dH \tag{2.7.9}$$

for some $\mathcal{C}^\infty$ functions, $H \in \mathcal{C}^\infty(U)$.

Let's see what this condition looks like in coordinates. Let

$$v = \sum f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i}\,. \tag{2.7.10}$$

Then

$$\begin{aligned} \iota(v)\omega &= \sum_{i,j} f_i \iota\left(\frac{\partial}{\partial x_i}\right) dx_j \wedge dy_j \\ &\quad + \sum_{i,j} g_i \iota\left(\frac{\partial}{\partial y_i}\right) dx_j \wedge dy_i\,. \end{aligned}$$

But

$$\iota\left(\frac{\partial}{\partial x_i}\right) dx_j = \begin{cases} 1 & i = i \\ 0 & i \neq j \end{cases}$$

and

$$\iota\left(\frac{\partial}{\partial x_i}\right) dy_j = 0$$

so the first summand above is

$$\sum f_i\, dy_i$$

and a similar argument shows that the second summand is

$$-\sum g_i\, dx_i\,.$$

Hence if v is the vector field (2.7.10)

$$\iota(v)\omega = \sum f_i\, dy_i - g_i\, dx_i\,. \tag{2.7.11}$$

Thus since

$$dH = \sum \frac{\partial H}{\partial x_i}\, dx_i + \frac{\partial H}{\partial y_i}\, dy_i$$

we get from (2.7.9)–(2.7.11)

$$f_i = \frac{\partial H}{\partial y_i} \text{ and } g_i = -\frac{\partial H}{\partial x_i} \tag{2.7.12}$$

so v has the form:

$$v = \sum \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}\,. \tag{2.7.13}$$

In particular if $\gamma(t) = (x(t)\,,\, y(t))$ is an integral curve of v it has to satisfy the system of differential equations

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial y_i}(x(t)\,,\, y(t)) \\ \frac{dy_i}{dt} &= -\frac{\partial H}{\partial x_i}(x(t)\,,\, y(t))\,. \end{aligned} \tag{2.7.14}$$

The formulas (2.7.10) and (2.7.11) exhibit an important property of the Darboux form, ω. Every one-form on U can be written uniquely as a sum

$$\sum f_i\, dy_i - g_i\, dx_i$$

with f_i and g_i in $\mathcal{C}^\infty(U)$ and hence (2.7.10) and (2.7.11) imply

Theorem 2.7.3. *The map, $v \to \iota(v)\omega$, sets up a* one-one correspondence *between vector field and one-forms.*

In particular for every $\mathcal{C}^\infty$ function, H, we get by correspondence a unique vector field, $v = v_H$, with the property (2.7.9).

We next note that by (1.7.6)

$$L_v H = \iota(v)\, dH = \iota(v)(\iota(v)\omega) = 0\,.$$

Thus

$$L_v H = 0 \tag{2.7.15}$$

i.e., H is an integral of motion of the vector field, v. In particular if the function, $H : U \to \mathbb{R}$, is proper, then by Theorem 2.1.10 the vector field, v, is complete and hence by Theorem 2.7.1 generates a one-parameter group of symplectomorphisms.

One last comment before we discuss the applications of these results to classical mechanics. If the one-parameter group (2.7.6) is a group of symplectomorphisms then $f_t^*\omega^n = f_t^*\omega \wedge \cdots \wedge f_t^*\omega = \omega^n$ so by (2.7.4)

$$f_t^*\Omega = \Omega \tag{2.7.16}$$

where Ω is the symplectic volume form (2.7.5).

The application we want to make of these ideas concerns the description, in Newtonian mechanics, of a physical system consisting of N interacting point-masses. The *configuration space* of such a system is

$$\mathbb{R}^n = \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \qquad (N \text{ copies})$$

with position coordinates, $x_1, \ldots, x_n$ and the *phase space* is $\mathbb{R}^{2n}$ with position coordinates $x_1, \ldots, x_n$ and momentum coordinates, $y_1, \ldots, y_n$. The *kinetic energy* of this system is a quadratic function of the momentum coordinates

$$\frac{1}{2}\sum \frac{1}{m_i} y_i^2\,, \tag{2.7.17}$$

and for simplicity we'll assume that the potential energy is a function, $V(x_1, \ldots, x_n)$, of the position coordinates alone, i.e., it doesn't depend on the momenta and is time-independent as well. Let

$$H = \frac{1}{2}\sum \frac{1}{m_i} y_i^2 + V(x_1, \ldots, x_n) \tag{2.7.18}$$

be the *total energy* of the system. We'll show below that Newton's second law of motion in classical mechanics reduces to the assertion: *the trajectories in phase space of the system above are just the integral curves of the Hamiltonian vector field,* v_H.

Proof. For the function (2.7.18) the equations (2.7.14) become

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{1}{m_i} y_i \\ \frac{dy_i}{dt} &= -\frac{\partial V}{\partial x_i}\,. \end{aligned} \tag{2.7.19}$$

The first set of equation are essentially just the definitions of momenta, however, if we plug them into the second set of equations we get

(2.7.20)
$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i}$$

and interpreting the term on the right as the force exerted on the i^{th} point-mass and the term on the left as mass times acceleration this equation becomes Newton's second law.

□

In classical mechanics the equations (2.7.14) are known as the Hamilton–Jacobi equations. For a more detailed account of their role in classical mechanics we highly recommend Arnold's book, *Mathematical Methods of Classical Mechanics*. Historically these equations came up for the first time, not in Newtonian mechanics, but in gemometric optics and a brief description of their origins there and of their relation to Maxwell's equations can be found in the bookl we cited above, *Symplectic Techniques in Physics*.

We'll conclude this chapter by mentioning a few implications of the Hamiltonian description (2.7.14) of Newton's equations (2.7.20).

1. *Conservation of energy.* By (2.7.15) the energy function (2.7.18) is constant along the integral curves of v, hence the energy of the system (2.7.14) doesn't change in time.

2. *Noether's principle.* Let $\gamma_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a one-parameter group of diffeomorphisms of phase space and w its infinitesmal generator. The γ_t's are called a *symmetry* of the system above if

(a) They preserve the function (2.7.18)
and
(b) the vector field w is Hamiltonian.

The condition (b) means that

(2.7.21)
$$\iota(w)\omega = dG$$

for some $\mathcal{C}^\infty$ function, G, and what Noether's principle asserts is that *this function is an integral of motion of the system* (2.7.14), i.e., satisfies $L_v G = 0$. In other words stated more succinctly: symmetries of the system (2.7.14) give rise to integrals of motion.

3. *Poincaré recurrence.* An important theorem of Poincaré asserts that if the function $H : \mathbb{R}^{2n} \to \mathbb{R}$ defined by (2.7.18) is proper then every trajectory of the system (2.7.14) returns arbitrarily close to its initial position at some positive time, t_0, and, in fact, does this not just once but does so infinitely often. We'll sketch a proof of this theorem, using (2.7.16), in the next chapter.

Exercises.

1. Let v_H be the vector field (2.7.13). Prove that $\text{div}(v_H) = 0$.

2. Let U be an open subset of $\mathbb{R}^m$, $f_t : U \to U$ a one-parameter group of diffeomorphisms of U and v the infinitesmal generator of this group. Show that if α is a k-form on U then $f_t^*\alpha = \alpha$ for all t if and only if $L_v\alpha = 0$ (i.e., generalize to arbitrary k-forms the result we proved above for the Darboux form).

3. The harmonic oscillator. Let H be the function $\sum_{i=1}^n m_i(x_i^2 + y_i^2)$ where the m_i's are positive constants.

(a) Compute the integral curves of v_H.

(b) Poincaré recurrence. Show that if $(x(t), y(t))$ is an integral curve with initial point $(x_0, y_0) = (x(0), y(0))$ and U an arbitrarily small neighborhood of (x_0, y_0), then for *every* $c > 0$ there exists a $t > c$ such that $(x(t), y(t)) \in U$.

4. Let U be an open subset of $\mathbb{R}^{2n}$ and let H_i, $i = 1, 2$, be in $\mathcal{C}^\infty(U)_i$. Show that

$$[v_{H_1}, v_{H_2}] = v_H \tag{2.7.22}$$

where

$$H = \sum_{i=1}^n \frac{\partial H_1}{\partial x_i}\frac{\partial H_2}{\partial y_i} - \frac{\partial H_2}{\partial x_i}\frac{\partial H_1}{\partial y_i} \,. \tag{2.7.23}$$

5. The expression (2.7.23) is known as the *Poisson bracket* of H_1 and H_2 and is denoted by $\{H_1, H_2\}$. Show that it is anti-symmetric

$$\{H_1, H_2\} = -\{H_2, H_1\}$$

and satisfies Jacobi's identity

$$0 = \{H_1, \{H_2, H_3\}\} + \{H_2, \{H_3, H_1\}\} + \{H_3, \{H_1, H_2\}\}\,.$$

6. Show that

$$\{H_1, H_2\} = L_{v_{H_1}} H_2 = -L_{v_{H_2}} H_1\,. \tag{2.7.24}$$

7. Prove that the following three properties are equivalent.

(a) $\{H_1, H_2\} = 0$.

(b) H_1 is an integral of motion of v_2.

(c) H_2 is an integral of motion of v_1.

8. Verify Noether's principle.

9. Conservation of linear momentum. Suppose the potential, V in (2.7.18) is invariant under the one-parameter group of translations

$$T_t(x_1, \ldots, x_n) = (x_1 + t, \ldots, x_n + t)\,.$$

(a) Show that the function (2.7.18) is invariant under the group of diffeomorphisms

$$\gamma_t(x, y) = (T_t x, y)\,.$$

(b) Show that the infinitesmal generator of this group is the Hamiltonian vector field v_G where $G = \sum_{i=1}^n y_i$.

(c) Conclude from Noether's principle that this function is an integral of the vector field v_H, i.e., that "total linear moment" is conserved.

(d) Show that "total linear momentum" is conserved if V is the Coulomb potential

$$\sum_{i \neq j} \frac{m_i}{|x_i - x_j|}\,.$$

10. Let $R_t^i : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the rotation which fixes the variables, (x_k, y_k), $k \neq i$ and rotates (x_i, y_i) by the angle, t:

$$R_t^i(x_i, y_i) = (\cos t\, x_i + \sin t\, y_i\,,\ -\sin t\, x_i + \cos t\, y_i)\,.$$

(a) Show that R^i_t, $-\infty < t < \infty$, is a one-parameter group of symplectomorphisms.

(b) Show that its generator is the Hamiltonian vector field, v_{H_i}, where $H_i = (x_i^2 + y_i^2)/2$.

(c) Let H be the "harmonic oscillator" Hamiltonian in exercise 3. Show that the R^j_t's preserve H.

(d) What does Noether's principle tell one about the classical mechanical system with energy function H?

11. Show that if U is an open subset of $\mathbb{R}^{2n}$ and v is a symplectic vector field on U then for every point, $p_0 \in U$, there exists a neighborhood, U_0, of p_0 on which v is Hamiltonian.

12. Deduce from exercises 4 and 11 that if v_1 and v_2 are symplectic vector fields on an open subset, U, of $\mathbb{R}^{2n}$ their Lie bracket, $[v_1, v_2]$, is a Hamiltonian vector field.

13. Let α be the one-form, $\sum_{i=1}^n y_i\, dx_i$.

(a) Show that $\omega = -\, d\alpha$.

(b) Show that if α_1 is any one-form on $\mathbb{R}^{2n}$ with the property, $\omega = -d\alpha_1$, then

$$\alpha = \alpha_1 + F$$

for some $\mathcal{C}^\infty$ function F.

(c) Show that $\alpha = \iota(w)\omega$ where w is the vector field

$$-\sum y_i \frac{\partial}{\partial y_i}\,.$$

14. Let U be an open subset of $\mathbb{R}^{2n}$ and v a vector field on U. Show that v has the property, $L_v\alpha = 0$, if and only if

$$\iota(v)\omega = d\iota(v)\alpha\,. \tag{2.7.25}$$

In particular conclude that if $L_v\alpha = 0$ then v is Hamiltonian. *Hint:* (2.7.2).

15. Let H be the function

$$H(x,y) = \sum f_i(x) y_i\,, \tag{2.7.26}$$

where the f_i's are $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$. Show that

$$L_{v_H}\alpha = 0\,. \tag{2.7.27}$$

16. Conversely show that if H is any $\mathcal{C}^\infty$ function on $\mathbb{R}^{2n}$ satisfying (2.7.27) it has to be a function of the form (2.7.26). *Hints:*

(a) Let v be a vector field on $\mathbb{R}^{2n}$ satisfying $L_v\alpha = 0$. By the previous exercise $v = v_H$, where $H = \iota(v)\alpha$.

(b) Show that H has to satisfy the equation

$$\sum_{i=1}^n y_i \frac{\partial H}{\partial y_i} = H\,.$$

(c) Conclude that if $H_r = \frac{\partial H}{\partial y_r}$ then H_r has to satisfy the equation

$$\sum_{i=1}^n y_i \frac{\partial}{\partial y_i} H_r = 0\,.$$

(d) Conclude that H_r has to be constant along the rays (x, ty), $0 \le t < \infty$.

(e) Conclude finally that H_r has to be a function of x alone, i.e., doesn't depend on y.

17. Show that if $v_{\mathbb{R}^n}$ is a vector field

$$\sum f_i(x) \frac{\partial}{\partial x_i}$$

on configuration space there is a unique lift of $v_{\mathbb{R}^n}$ to phase space

$$v = \sum f_i(x) \frac{\partial}{\partial x_i} + g_i(x, y) \frac{\partial}{\partial y_i}$$

satisfying $L_v\alpha = 0$.

CHAPTER 3

INTEGRATION OF FORMS

3.1 Introduction

The change of variables formula asserts that if U and V are open subsets of $\mathbb{R}^n$ and $f : U \to V$ a C^1 diffeomorphism then, for every continuous function, $\varphi : V \to \mathbb{R}$ the integral

$$\int_V \varphi(y)\, dy$$

exists if and only if the integral

$$\int_U \varphi \circ f(x) |\det Df(x)|\, dx$$

exists, and if these integrals exist they are equal. Proofs of this can be found in [**?**], [**?**] or [**?**]. This chapter contains an alternative proof of this result. This proof is due to Peter Lax. Our version of his proof in §3.5 below makes use of the theory of differential forms; but, as Lax shows in the article [**?**] (which we strongly recommend as collateral reading for this course), references to differential forms can be avoided, and the proof described in§3.5 can be couched entirely in the language of elementary multivariable calculus.

The virtue of Lax's proof is that is allows one to prove a version of the change of variables theorem for other mappings besides diffeomorphisms, and involves a topological invariant, the *degree of a mapping*, which is itself quite interesting. Some properties of this invariant, and some topological applications of the change of variables formula will be discussed in §3.6 of these notes.

Remark 3.1.1. *The proof we are about to describe is somewhat simpler and more transparent if we assume that f is a $\mathcal{C}^\infty$ diffeomorphism. We'll henceforth make this assumption.*

3.2 The Poincaré lemma for compactly supported forms on rectangles

Let ν be a k-form on $\mathbb{R}^n$. We define the *support* of ν to be the closure of the set

$$\{x \in \mathbb{R}^n \,,\ \nu_x \neq 0\}$$

and we say that ν is *compactly* supported if $\operatorname{supp} \nu$ is compact. We will denote by $\Omega_c^k(\mathbb{R}^n)$ the set of all $\mathcal{C}^\infty$ k-forms which are compactly supported, and if U is an open subset of $\mathbb{R}^n$, we will denote by $\Omega_c^k(U)$ the set of all compactly supported k-forms whose support is contained in U .

Let $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ be a compactly supported n-form with $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We will define the integral of ω over $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \omega$$

to be the usual integral of f over $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} f\, dx\,.$$

(Since f is $\mathcal{C}^\infty$ and compactly supported this integral is well-defined.)

Now let Q be the rectangle

$$[a_1, b_1] \times \cdots \times [a_n, b_n]\,.$$

The Poincaré lemma for rectangles asserts:

Theorem 3.2.1. *Let ω be a compactly supported n-form, with* $\operatorname{supp} \omega \subseteq \operatorname{Int} Q$*. Then the following assertions are equivalent:*

a. $\int \omega = 0$.

b. *There exists a compactly supported $(n-1)$-form, μ, with* $\operatorname{supp} \mu \subseteq \operatorname{Int} Q$ *satisfying $d\mu = \omega$.*

We will first prove that (b)⇒(a). Let

$$\mu = \sum_{i=1}^{n} f_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\,,$$

(the "hat" over the dx_i meaning that dx_i has to be omitted from the wedge product). Then

$$d\mu = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n \,,$$

and to show that the integral of $d\mu$ is zero it suffices to show that each of the integrals

$$(2.1)_i \qquad \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}\, dx$$

is zero. By Fubini we can compute $(2.1)_i$ by first integrating with respect to the variable, x_i, and then with respect to the remaining variables. But

$$\int \frac{\partial f}{\partial x_i}\, dx_i = f(x) \Big|_{x_i=a_i}^{x_i=b_i} = 0$$

since f_i is supported on U.

We will prove that (a) $\Rightarrow$ (b) by proving a somewhat stronger result. Let U be an open subset of $\mathbb{R}^m$. We'll say that U has *property* P if every form, $\omega \in \Omega_c^m(U)$ whose integral is zero in $d\,\Omega_c^{m-1}(U)$.

We will prove

Theorem 3.2.2. *Let U be an open subset of $\mathbb{R}^{n-1}$ and $A \subseteq \mathbb{R}$ an open interval. Then if U has property P, $U \times A$ does as well.*

Remark 3.2.3. *It's very easy to see that the open interval A itself has property P. (See exercise 1 below.) Hence it follows by induction from Theorem 3.2.2 that*

$$\operatorname{Int} Q = A_1 \times \cdots \times A_n, \quad A_i = (a_i, b_i)$$

has property P, and this proves "(a) $\Rightarrow$ (b)".

To prove Theorem 3.2.2 let $(x,t) = (x_1, \ldots, x_{n-1}, t)$ be product coordinates on $U \times A$. Given $\omega \in \Omega_c^n(U \times A)$ we can express ω as a wedge product, $dt \wedge \alpha$ with $\alpha = f(x,t)\, dx_1 \wedge \cdots \wedge dx_{n-1}$ and $f \in \mathcal{C}_0^\infty(U \times A)$. Let $\theta \in \Omega_c^{n-1}(U)$ be the form

$$(3.2.1) \qquad \theta = \left(\int_A f(x,t)\, dt \right) dx_1 \wedge \cdots \wedge dx_{n-1} \,.$$

Then

$$\int_{\mathbb{R}^{n-1}} \theta = \int_{\mathbb{R}^n} f(x,t)\, dx\, dt = \int_{\mathbb{R}^n} \omega$$

so if the integral of ω is zero, the integral of θ is zero. Hence since U has property P, $\beta = d\nu$ for some $\nu \in \Omega_c^{n-1}(U)$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a bump function which is supported on A and whose integral over A is one. Setting

$$\kappa = -\rho(t)\, dt \wedge \nu$$

we have

$$d\kappa = \rho(t)\, dt \wedge\, d\nu = \rho(t)\, dt \wedge \theta\,,$$

and hence

$$\omega - d\kappa \quad = \quad dt \wedge (\alpha - \rho(t)\theta) = dt \wedge u(x,t)\, dx_1 \wedge \cdots \wedge\, dx_{n-1}$$

where

$$u(x,t) \quad = \quad f(x,t) - \rho(t) \int_A f(x,t)\, dt$$

by (3.2.1). Thus

$$\int u(x,t)\, dt = 0\,. \tag{3.2.2}$$

Let a and b be the end points of A and let

$$v(x,t) = \int_a^t i(x,s)\, ds\,. \tag{3.2.3}$$

By (3.2.2) $v(a,x) = v(b,x) = 0$, so v is in $\mathcal{C}_0^\infty(U \times A)$ and by (3.2.3), $\partial v/\partial t = u$. Hence if we let γ be the form, $v(x,t)\, dx_1 \wedge \cdots \wedge dx_{n-1}$, we have:

$$d\gamma \quad = \quad u(x,t)\, dx \wedge \cdots \wedge dx_{n-1} = \omega - d\kappa$$

and

$$\omega \quad = \quad d(\gamma + \kappa)\,.$$

Since γ and κ are both in $\Omega_c^{n-1}(U \times A)$ this proves that ω is in $d\,\Omega_c^{n-1}(U \times A)$ and hence that $U \times A$ has property P.

Exercises for §3.2.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class C^r with support on the interval, (a,b). Show that the following are equivalent.

(a) $\int_a^b f(x)\,dx = 0$.

(b) There exists a function, $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) with $\frac{dg}{dx} = f$.

Hint: Show that the function

$$g(x) = \int_a^x f(s)\,ds$$

is compactly supported.

2. Let $f = f(x, y)$ be a compactly supported function on $\mathbb{R}^k \times \mathbb{R}^\ell$ with the property that the partial derivatives

$$\frac{\partial f}{\partial x_i}(x, y)\,,\ i = 1, \ldots, k\,,$$

and are continuous as functions of x and y. Prove the following "differentiation under the integral sign" theorem (which we implicitly used in our proof of Theorem 3.2.2).

Theorem 3.2.4. *The function*

$$g(x) = \int f(x, y)\,dy$$

is of class C^1 and

$$\frac{\partial g}{\partial x_i}(x) = \int \frac{\partial f}{\partial x_i}(x, y)\,dy\,.$$

Hints: For y fixed and $h \in \mathbb{R}^k$,

$$f_i(x + h, y) - f_i(x, y) = D_x f_i(c)h$$

for some point, c, on the line segment joining x to $x + c$. Using the fact that $D_x f$ is continuous as a function of x and y and compactly supported, conclude:

Lemma 3.2.5. *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for $|h| \le \delta$*

$$|f(x + h, y) - f(x, y) - D_x f(x, c)h| \le \epsilon |h|\,.$$

Now let $Q \subseteq \mathbb{R}^\ell$ be a rectangle with $\operatorname{supp} f \subseteq \mathbb{R}^k \times Q$ and show that

$$|g(x+h) - g(x) - \left(\int D_x f(x,y)\,dy\right) h| \leq \epsilon \operatorname{vol}(Q)|h|\,.$$

Conclude that g is differentiable at x and that its derivative is

$$\int D_x f(x,y)\,dy\,.$$

3. Let $f : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}$ be a compactly supported continuous function. Prove

Theorem 3.2.6. *If all the partial derivatives of $f(x,y)$ with respect to x of order $\leq r$ exist and are continuous as functions of x and y the function*

$$g(x) = \int f(x,y)\,dy$$

is of class C^r.

4. Let U be an open subset of $\mathbb{R}^{n-1}$, $A \subseteq \mathbb{R}$ an open interval and (x,t) product coordinates on $U \times A$. Recall (§2.2) exercise 5) that every form, $\omega \in \Omega^k(U \times A)$, can be written uniquely as a sum, $\omega = dt \wedge \alpha + \beta$ where α and β are *reduced*, i.e., don't contain a factor of dt.

(a) Show that if ω is compactly supported on $U \times A$ then so are α and β.

(b) Let $\alpha = \sum_I f_I(x,t)\,dx_I$. Show that the form

$$\theta = \sum_I \left(\int_A f_I(x,t)\,dt\right) dx_I \tag{3.2.4}$$

is in $\Omega_c^{k-1}(U)$.

(c) Show that if $d\omega = 0$, then $d\theta = 0$. *Hint:* By (3.2.4)

$$\begin{aligned} d\theta &= \sum_{I,i} \left(\int_A \frac{\partial f_I}{\partial x_i}(x,t)\,dt\right) dx_i \wedge dx_I \\ &= \int_A (d_U \alpha)\,dt \end{aligned}$$

and by (**??**) $d_U \alpha = \dfrac{d\beta}{dt}$.

5. In exercise 4 show that if θ is in $d\,\Omega^{k-1}(U)$ then ω is in $d\,\Omega^k_c(U)$.
Hints:

(a) Let $\theta = d\nu$, with $\nu = \Omega^{k-2}_c(U)$ and let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a bump function which is supported on A and whose integral over A is one. Setting $k = -\rho(t)\,dt \wedge \nu$ show that

$$\begin{aligned} \omega - d\kappa &= dt \wedge (\alpha - \rho(t)\theta) + \beta \\ &= dt \wedge (\sum_I u_I(x,t)\,dx_I) + \beta \end{aligned}$$

where

$$u_I(x,t) = f_I(x,t) - \rho(t)\int_A f_I(x,t)\,dt\,.$$

(b) Let a and b be the end points of A and let

$$v_I(x,t) = \int_a^t u_I(x,t)\,dt\,.$$

Show that the form $\sum v_I(x,t)\,dx_I$ is in $\Omega^{k-1}_c(U \times A)$ and that

$$d\gamma = \omega - d\kappa - \beta - d_U\gamma\,.$$

(c) Conclude that the form $\omega - d(\kappa + \gamma)$ is reduced.

(d) Prove: If $\lambda \in \Omega^k_c(U \times A)$ is reduced and $d\lambda = 0$ then $\lambda = 0$.
Hint: Let $\lambda = \sum g_I(x,t)\,dx_I$. Show that $d\lambda = 0 \Rightarrow \frac{\partial}{\partial t} g_I(x,t) = 0$ and exploit the fact that for fixed x, $g_I(x,t)$ is compactly supported in t.

6. Let U be an open subset of $\mathbb{R}^m$. We'll say that U has *property* P_k, for $k < n$, if every closed k-form, $\omega \in \Omega^k_c(U)$, is in $d\,\Omega^{k-1}_c(U)$. Prove that if the open set $U \subseteq \mathbb{R}^{n-1}$ in exercise 3 has property P_k then so does $U \times A$.

7. Show that if Q is the rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ and $U = \operatorname{Int} Q$ then u has property P_k.

8. Let $\mathbb{H}^n$ be the half-space

$$\{(x_1, \ldots, x_n)\,; \quad x_1 \le 0\} \tag{3.2.5}$$

and let $\omega \in \Omega^n_c(\mathbb{R})$ be the n-form, $f\,dx_1 \wedge \cdots \wedge dx_n$ with $f \in \mathcal{C}^\infty_0(\mathbb{R}^n)$. Define:

$$\int_{\mathbb{H}^n} \omega = \int_{\mathbb{H}^n} f(x_1, \ldots, x_n)\,dx_1 \cdots dx_n \tag{3.2.6}$$

where the right hand side is the usual Riemann integral of f over $\mathbb{H}^n$. (This integral makes sense since f is compactly supported.) Show that if $\omega = d\mu$ for some $\mu \in \Omega^{n-1}_c(\mathbb{R}^n)$ then

$$\int_{\mathbb{H}^n} \omega = \int_{\mathbb{R}^{n-1}} \iota^*\mu \tag{3.2.7}$$

where $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is the inclusion map

$$(x_2, \ldots, x_n) \to (0, x_2, \ldots, x_n)\,.$$

Hint: Let $\mu = \sum_i f_i\,dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$. Mimicking the "(b) $\Rightarrow$ (a)" part of the proof of Theorem 3.2.1 show that the integral (3.2.6) is the integral over $\mathbb{R}^{n-1}$ of the function

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}(x_1, x_2, \ldots, x_n)\,dx_1\,.$$

3.3 The Poincaré lemma for compactly supported forms on open subsets of $\mathbb{R}^n$

In this section we will generalize Theorem 3.2.1 to arbitrary connected open subsets of $\mathbb{R}^n$.

Theorem 3.3.1. *Let U be a connected open subset of $\mathbb{R}^n$ and let ω be a compactly supported n-form with* $\operatorname{supp}\omega \subset U$. *The the following assertions are equivalent,*

a. $\int \omega = 0$.

b. *There exists a compactly supported $(n-1)$-form, μ, with* $\operatorname{supp}\mu \subseteq U$ *and $\omega = d\mu$.*

Proof that (b) $\Rightarrow$ (a). The support of μ is contained in a large rectangle, so the integral of $d\mu$ is zero by Theorem 3.2.1.

Proof that (a) $\Rightarrow$ (b): Let ω_1 and ω_2 be compactly supported n-forms with support in U. We will write

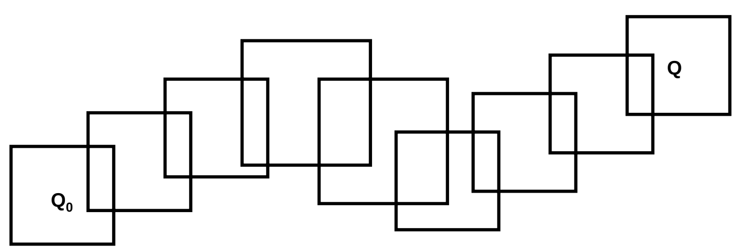
$$\omega_1 \sim \omega_2$$

as shorthand notation for the statement: "There exists a compactly supported $(n-1)$-form, μ, with support in U and with $\omega_1 - \omega_2 = d\mu$.", We will prove that (a) $\Rightarrow$ (b) by proving an equivalent statement: Fix a rectangle, $Q_0 \subset U$ and an n-form, ω_0, with $\operatorname{supp} \omega_0 \subseteq Q_0$ and integral equal to one.

Theorem 3.3.2. *If ω is a compactly supported n-form with* $\operatorname{supp} \omega \subseteq U$ *and $c = \int \omega$ then $\omega \sim c\omega_0$.*

Thus in particular if $c = 0$, Theorem 3.3.2 says that $\omega \sim 0$ proving that (a) $\Rightarrow$ (b).

To prove Theorem 3.3.2 let $Q_i \subseteq U$, $i = 1, 2, 3, \ldots$, be a collection of rectangles with $U = \cup \operatorname{Int} Q_i$ and let φ_i be a partition of unity with $\operatorname{supp} \varphi_i \subseteq \operatorname{Int} Q_i$. Replacing ω by the finite sum $\sum_{i=1}^{m} \varphi_i \omega$, m large, it suffices to prove Theorem 3.3.2 for each of the summands $\varphi_i \omega$. In other words we can assume that $\operatorname{supp} \omega$ is contained in one of the open rectangles, $\operatorname{Int} Q_i$. Denote this rectangle by Q. We claim that one can join Q_0 to Q by a sequence of rectangles as in the figure below.



Lemma 3.3.3. *There exists a sequence of rectangles, R_i, $i = 0, \ldots, N+1$ such that $R_0 = Q_0$, $R_{N+1} = Q$ and* $\operatorname{Int} R_i \cap \operatorname{Int} R_{i+1}$ *is non-empty.*

Proof. Denote by A the set of points, $x \in U$, for which there exists a sequence of rectangles, R_i, $i = 0, \ldots, N+1$ with $R_0 = Q_0$, with $x \in \operatorname{Int} R_{N+1}$ and with $\operatorname{Int} R_i \cap \operatorname{Int} R_{i+1}$ non-empty. It is clear that this set is open and that its complement is open; so, by the connectivity of U, $U = A$. □

To prove Theorem 3.3.2 with $\operatorname{supp}\omega \subseteq Q$, select, for each i, a compactly supported n-form, ν_i, with $\operatorname{supp}\nu_i \subseteq \operatorname{Int} R_i \cap \operatorname{Int} R_{i+1}$ and with $\int \nu_i = 1$. The difference, $\nu_i - \nu_{i+1}$ is supported in $\operatorname{Int} R_{i+1}$, and its integral is zero; so by Theorem 3.2.1, $\nu_i \sim \nu_{i+1}$. Similarly, $\omega_0 \sim \nu_1$ and, if $c = \int \omega$, $\omega \sim c\nu_N$. Thus

$$c\omega_0 \sim c\nu_0 \sim \cdots \sim c\nu_N = \omega$$

proving the theorem.

3.4 The degree of a differentiable mapping

Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$. A continuous mapping, $f : U \to V$, is *proper* if, for every compact subset, B, of V, $f^{-1}(B)$ is compact. Proper mappings have a number of nice properties which will be investigated in the exercises below. One obvious property is that if f is a $\mathcal{C}^\infty$ mapping and ω is a compactly supported k-form with support on V, $f^*\omega$ is a compactly supported k-form with support on U. Our goal in this section is to show that if U and V are connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a proper $\mathcal{C}^\infty$ mapping then there exists a topological invariant of f, which we will call its *degree* (and denote by $\deg(f)$), such that the "change of variables" formula:

$$\int_U f^*\omega = \deg(f) \int_V \omega \tag{3.4.1}$$

holds for all $\omega \in \Omega^n_c(V)$.

Before we prove this assertion let's see what this formula says in coordinates. If

$$\omega = \varphi(y)\, dy_1 \wedge \cdots \wedge dy_n$$

then at $x \in U$

$$f^*\omega = (\varphi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n\,;$$

so, in coordinates, (3.4.1) takes the form

$$\int_V \varphi(y)\, dy = \deg(f) \int_U \varphi \circ f(x) \det(Df(x))\, dx\,. \tag{3.4.2}$$

Proof of 3.4.1. Let ω_0 be an n-form of compact support with $\operatorname{supp}\omega_0 \subset V$ and with $\int \omega_0 = 1$. If we set $\deg f = \int_U f^*\omega_0$ then (3.4.1) clearly holds for ω_0. We will prove that (3.4.1) holds for every compactly supported n-form, ω, with $\operatorname{supp}\omega \subseteq V$. Let $c = \int_V \omega$. Then by Theorem 3.1 $\omega - c\omega_0 = d\mu$, where μ is a completely supported $(n-1)$-form with $\operatorname{supp}\mu \subseteq V$. Hence

$$f^*\omega - cf^*\omega_0 = f^*\, d\mu = d\, f^*\mu\,,$$

and by part (a) of Theorem 3.1

$$\int_U f^*\omega = c\int f^*\omega_0 = \deg(f)\int_V \omega\,.$$

□

We will show in § 3.6 that the degree of f is always an integer and explain why it is a "topological" invariant of f. For the moment, however, we'll content ourselves with pointing out a simple but useful property of this invariant. Let U, V and W be connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ and $g : V \to W$ proper $\mathcal{C}^\infty$ mappings. Then

(3.4.3) $$\deg(g \circ f) = \deg(g)\deg(f)\,.$$

Proof. Let ω be a compactly supported n-form with support on W. Then

$$(g \circ f)^*\omega = g^* f^*\omega\,;$$

so

$$\begin{aligned}\int_U (g\circ f)^*\omega &= \int_U g^*(f^*\omega) = \deg(g)\int_V f^*\omega \\ &= \deg(g)\deg(f)\int_W \omega\,.\end{aligned}$$

□

From this multiplicative property it is easy to deduce the following result (which we will need in the next section).

Theorem 3.4.1. *Let A be a non-singular $n \times n$ matrix and $f_A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with A. Then $\deg(f_A) = +1$ if $\det A$ is positive and -1 if $\det A$ is negative.*

A proof of this result is outlined in exercises 5–9 below.

Exercises for §3.4.

1. Let U be an open subset of $\mathbb{R}^n$ and φ_i, $i = 1, 2, 3, \ldots$, a partition of unity on U. Show that the mapping, $f : U \to \mathbb{R}$ defined by

$$f = \sum_{k=1}^{\infty} k\varphi_k$$

is a proper $\mathcal{C}^\infty$ mapping.

2. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$ and let $f : U \to V$ be a proper continuous mapping. Prove:

Theorem 3.4.2. *If B is a compact subset of V and $A = f^{-1}(B)$ then for every open subset, U_0, with $A \subseteq U_0 \subseteq U$, there exists an open subset, V_0, with $B \subseteq V_0 \subseteq V$ and $f^{-1}(V_0) \subseteq U_0$.*

Hint: Let C be a compact subset of V with $B \subseteq \operatorname{Int} C$. Then the set, $W = f^{-1}(C) - U_0$ is compact; so its image, $f(W)$, is compact. Show that $f(W)$ and B are disjoint and let

$$V_0 = \operatorname{Int} C - f(W)\,.$$

3. Show that if $f : U \to V$ is a proper continuous mapping and X is a closed subset of U, $f(X)$ is closed.

Hint: Let $U_0 = U - X$. Show that if p is in $V - f(X)$, $f^{-1}(p)$ is contained in U_0 and conclude from the previous exercise that there exists a neighborhood, V_0, of p such that $f^{-1}(V_0)$ is contained in U_0. Conclude that V_0 and $f(X)$ are disjoint.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $f(x) = x + a$. Show that $\deg(f) = 1$.

Hint: Let $\psi : \mathbb{R} \to \mathbb{R}$ be a compactly supported $\mathcal{C}^\infty$ function. For $a \in \mathbb{R}$, the identity

$$\int \psi(t)\,dt = \int \psi(t-a)\,dt \tag{3.4.4}$$

is easy to prove by elementary calculus, and this identity proves the assertion above in dimension one. Now let

$$\varphi(x) = \psi(x_1)\ldots\varphi(x_n) \tag{3.4.5}$$

and compute the right and left sides of (3.4.2) by Fubini's theorem.

5. Let σ be a permutation of the numbers, $1, \dots, n$ and let $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism, $f_\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Prove that $\deg f_\sigma = \operatorname{sgn}(\sigma)$.

Hint: Let φ be the function (3.4.5). Show that if ω is equal to $\varphi(x)\, dx_1 \wedge \cdots \wedge dx_n$, $f^*\omega = (\operatorname{sgn} \sigma)\omega$.

6. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (x_1 + \lambda x_2, x_2, \dots, x_n).$$

Prove that $\deg(f) = 1$.

Hint: Let $\omega = \varphi(x_1, \dots, x_n)\, dx_1 \wedge \ldots \wedge dx_n$ where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is compactly supported and of class $\mathcal{C}^\infty$. Show that

$$\int f^*\omega = \int \varphi(x_1 + \lambda x_2, x_2, \dots, x_n)\, dx_1 \dots dx_n$$

and evaluate the integral on the right by Fubini's theorem; i.e., by first integrating with respect to the x_1 variable and then with respect to the remaining variables. Note that by (3.4.4)

$$\int f(x_1 + \lambda x_2, x_2, \dots, x_n)\, dx_1 = \int f(x_1, x_2, \dots, x_n)\, dx_1 \,.$$

7. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (\lambda x_1, x_2, \dots, x_n)$$

with $\lambda \neq 0$. Show that $\deg f = +1$ if λ is positive and -1 if λ is negative.

Hint: In dimension 1 this is easy to prove by elementary calculus techniques. Prove it in d-dimensions by the same trick as in the previous exercise.

8. (a) Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{R}^n$ and A, B and C the linear mappings

$$\begin{aligned} Ae_1 &= e, \quad Ae_i = \sum_j a_{j,i} e_j \,, \qquad i > 1 \\ Be_i &= e_i \,, \quad i > 1 \,, \quad Be_1 = \sum_{j=1}^n b_j e_j \\ Ce_1 &= e_1 \,, \quad Ce_i = e_i + c_i e_1 \,, \quad i > 1 \,. \end{aligned} \tag{3.4.6}$$

Show that

$$BACe_1 = \sum b_j e_j$$

and

$$BACe_i = \sum_j^n = (a_{j,i} + c_i b_j)e_j + c_i b_1 e_1$$

for $i > 1$.

(b)

$$Le_i = \sum_{j=1}^n \ell_{j,i} e_j \,, \quad i = 1, \ldots, n \,. \tag{3.4.7}$$

Show that if $\ell_{1,1} \neq 0$ one can write L as a product, $L = BAC$, where A, B and C are linear mappings of the form (3.4.6).

Hint: First solve the equations

$$\ell_{j,1} = b_j$$

for $j = 1, \ldots, n$, then the equations

$$\ell_{1,i} = b_1 c_i$$

for $i > 1$, then the equations

$$\ell_{j,i} = a_{j,i} + c_i b_j$$

for $i, j > 1$.

(c) Suppose L is invertible. Conclude that A, B and C are invertible and verify that Theorem 3.4.1 holds for B and C using the previous exercises in this section.

(d) Show by an inductive argument that Theorem 3.4.1 holds for A and conclude from (3.4.3) that it holds for L.

9. To show that Theorem 3.4.1 holds for an arbitrary linear mapping, L, of the form (3.4.7) we'll need to eliminate the assumption: $\ell_{1,1} \neq 0$. Show that for some j, $\ell_{j,1}$ is non-zero, and show how to eliminate this assumption by considering $f_\sigma \circ L$ where σ is the transposition, $1 \leftrightarrow j$.

10. Here is an alternative proof of Theorem 4.3.1 which is shorter than the proof outlined in exercise 9 but uses some slightly more sophisticated linear algebra.

(a) Prove Theorem 3.4.1 for linear mappings which are *orthogonal*, i.e., satisfy $L^tL = I$.

Hints:

i. Show that $L^*(x_1^2 + \cdots + x_n^2) = x_1^2 + \cdots + x_n^2$.

ii. Show that $L^*(dx_1 \wedge \cdots \wedge dx_n)$ is equal to $dx_1 \wedge \cdots \wedge dx_n$ or $-dx_1 \wedge \cdots \wedge dx_n$ depending on whether L is orientation preserving or orinetation reversing. (See § 1.2, exercise 10.)

iii. Let ψ be as in exercise 4 and let ω be the form

$$\omega = \psi(x_1^2 + \cdots + x_n^2)\, dx_1 \wedge \cdots \wedge\, dx_n\,.$$

Show that $L^*\omega = \omega$ if L is orientation preserving and $L^*\omega = -\omega$ if L is orientation reversing.

(b) Prove Theorem 3.4.1 for linear mappings which are *self-adjoint* (satisfy $L^t = L$). *Hint:* A self-adjoint linear mapping is diagonizable: there exists an intervertible linear mapping, $M : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$M^{-1}LMe_i = \lambda_i e_i\,, \quad i = 1, \ldots, n\,. \tag{3.4.8}$$

(c) Prove that every invertible linear mapping, L, can be written as a product, $L = BC$ where B is orthogonal and C is self-adjoint.

Hints:

i. Show that the mapping, $A = L^tL$, is self-adjoint and that it's eigenvalues, the λ_i's in 3.4.8, are positive.

ii. Show that there exists an invertible self-adjoint linear mapping, C, such that $A = C^2$ and $AC = CA$.

iii. Show that the mapping $B = LC^{-1}$ is orthogonal.

3.5 The change of variables formula

Let U and V be connected open subsets of $\mathbb{R}^n$. If $f : U \to V$ is a diffeomorphism, the determinant of $Df(x)$ at $x \in U$ is non-zero, and hence, since it is a continuous function of x, its sign is the same at every point. We will say that f is *orientation preserving* if this sign is positive and *orientation reversing* if it is negative. We will prove below:

Theorem 3.5.1. *The degree of f is $+1$ if f is orientation preserving and -1 if f is orientation reversing.*

We will then use this result to prove the following change of variables formula for diffeomorphisms.

Theorem 3.5.2. *Let $\varphi : V \to \mathbb{R}$ be a compactly supported continuous function. Then*

$$\int_U \varphi \circ f(x) |\det(Df)(x)| = \int_V \varphi(y)\, dy \,. \tag{3.5.1}$$

Proof of Theorem 3.5.1. Given a point, $a_1 \in U$, let $a_2 = -f(a_1)$ and for $i = 1, 2$, let $g_i : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $g_i(x) = x + a_i$. By (3.4.1) and exercise 4 of § 4 the composite diffeomorphism

$$g_2 \circ f \circ g_1 \tag{3.5.2}$$

has the same degree as f, so it suffices to prove the theorem for this mapping. Notice however that this mapping maps the origin onto the origin. Hence, replacing f by this mapping, we can, without loss of generality, assume that 0 is in the domain of f and that $f(0) = 0$.

Next notice that if $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear mapping the theorem is true for A (by exercise 9 of § 3.4), and hence if we can prove the theorem for $A^{-1} \circ f$, (3.4.1) will tell us that the theorem is true for f. In particular, letting $A = Df(0)$, we have

$$D(A^{-1} \circ f)(0) = A^{-1} Df(0) = I$$

where I is the identity mapping. Therefore, replacing f by $A^{-1}f$, we can assume that the mapping, f, for which we are attempting to prove Theorem 3.5.1 has the properties: $f(0) = 0$ and $Df(0) = I$. Let $g(x) = f(x) - x$. Then these properties imply that $g(0) = 0$ and $Dg(0) = 0$.

□

Lemma 3.5.3. *There exists a $\delta > 0$ such that $|g(x)| \leq \frac{1}{2}|x|$ for $|x| \leq \delta$.*

Proof. Let $g(x) = (g_1(x), \ldots, g_n(x))$. Then

$$\frac{\partial g_i}{\partial x_j}(0) = 0\,;$$

so there exists a $\delta > 0$ such that

$$\left|\frac{\partial g_i}{\partial x_j}(x)\right| \le \frac{1}{2}$$

for $|x| \le \delta$. However, by the mean value theorem,

$$g_i(x) = \sum \frac{\partial g_i}{\partial x_j}(c)x_j$$

for $c = t_0 x$, $0 < t_0 < 1$. Thus, for $|x| < \delta$,

$$|g_i(x)| \le \frac{1}{2} \sup |x_i| = \frac{1}{2} |x| \,,$$

so

$$|g(x)| = \sup |g_i(x)| \le \frac{1}{2} |x| \,.$$

□

Let ρ be a compactly supported $\mathcal{C}^\infty$ function with $0 \le \rho \le 1$ and with $\rho(x) = 0$ for $|x| \ge \delta$ and $\rho(x) = 1$ for $|x| \le \frac{\delta}{2}$ and let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$\widetilde{f}(x) = x + \rho(x)g(x) \,. \tag{3.5.3}$$

It's clear that

$$\widetilde{f}(x) = x \text{ for } |x| \ge \delta \tag{3.5.4}$$

and, since $f(x) = x + g(x)$,

$$\widetilde{f}(x) = f(x) \text{ for } |x| \le \frac{\delta}{2} \,. \tag{3.5.5}$$

In addition, for all $x \in \mathbb{R}^n$:

$$|\widetilde{f}(x)| \ge \frac{1}{2} |x| \,. \tag{3.5.6}$$

Indeed, by (3.5.4), $|\widetilde{f}(x)| \ge |x|$ for $|x| \ge \delta$, and for $|x| \le \delta$

$$\begin{aligned} |\widetilde{f}(x)| &\ge |x| - \rho(x)|g(x)| \\ &\ge |x| - |g(x)| \ge |x| - \frac{1}{2} |x| = \frac{1}{2} |x| \end{aligned}$$

by Lemma 3.5.3.

Now let $\mathcal{Q}_r$ be the cube, $\{x \in \mathbb{R}^n \,,\ |x| \leq r\}$, and let $\mathcal{Q}_r^c = \mathbb{R}^n - \mathcal{Q}_r$. From (3.5.6) we easily deduce that

$$\widetilde{f}^{-1}(\mathcal{Q}_r) \subseteq \mathcal{Q}_{2r} \tag{3.5.7}$$

for all r, and hence that $\widetilde{f}$ is *proper*. Also notice that for $x \in \mathcal{Q}_\delta$,

$$|\widetilde{f}(x)| \leq |x| + |g(x)| \leq \frac{3}{2}\,|x|$$

by Lemma 3.5.3 and hence

$$\widetilde{f}^{-1}(\mathcal{Q}^c_{\frac{3}{2}\delta}) \subseteq \mathcal{Q}^c_\delta\,. \tag{3.5.8}$$

We will now prove Theorem 3.5.1. Since f is a diffeomorphism mapping 0 to 0, it maps a neighborhood, U_0, of 0 in U diffeomorphically onto a neighborhood, V_0, of 0 in V, and by shrinking U_0 if necessary we can assume that U_0 is contained in $\mathcal{Q}_{\delta/2}$ and V_0 contained in $\mathcal{Q}_{\delta/4}$. Let ω be an n-form with support in V_0 whose integral over $\mathbb{R}^n$ is equal to one. Then $f^*\omega$ is supported in U_0 and hence in $\mathcal{Q}_{\delta/2}$. Also by (3.5.7) $\widetilde{f}^*\omega$ is supported in $\mathcal{Q}_{\delta/2}$. Thus both of these forms are zero outside $\mathcal{Q}_{\delta/2}$. However, on $\mathcal{Q}_{\delta/2}$, $\widetilde{f} = f$ by (3.5.5), so these forms are equal everywhere, and hence

$$\deg(f) = \int f^*\omega = \int \widetilde{f}^*\omega = \deg(\widetilde{f})\,.$$

Next let ω be a compactly supported n-form with support in $\mathcal{Q}^c_{3\delta/2}$ and with integral equal to one. Then $\widetilde{f}^*\omega$ is supported in $\mathcal{Q}^c_\delta$ by (3.5.8), and hence since $f(x) = x$ on $\mathcal{Q}^c_\delta$ $\widetilde{f}^*\omega = \omega$. Thus

$$\deg(\widetilde{f}) = \int f^*\omega = \int \omega = 1\,.$$

Putting these two identities together we conclude that $\deg(f) = 1$. Q.E.D.

If the function, φ, in Theorem 3.5.2 is a $\mathcal{C}^\infty$ function, the identity (3.5.1) is an immediate consequence of the result above and the identity (3.4.2). If φ is not $\mathcal{C}^\infty$, but is just continuous, we will deduce Theorem 3.5.2 from the following result.

Theorem 3.5.4. *Let V be an open subset of $\mathbb{R}^n$. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function of compact support with* $\operatorname{supp}\varphi \subseteq V$*; then for every $\epsilon > 0$ there exists a $\mathcal{C}^\infty$ function of compact support, $\psi : \mathbb{R}^n \to \mathbb{R}$ with* $\operatorname{supp}\psi \subseteq V$ *and*

$$\sup|\psi(x) - \varphi(x)| < \epsilon\,.$$

Proof. Let A be the support of φ and let d be the distance in the sup norm from A to the complement of V. Since φ is continuous and compactly supported it is uniformly continuous; so for every $\epsilon > 0$ there exists a $\delta > 0$ with $\delta < \frac{d}{2}$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ when $|x - y| \le \delta$. Now let Q be the cube: $|x| < \delta$ and let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a non-negative $\mathcal{C}^\infty$ function with $\operatorname{supp}\rho \subseteq Q$ and

$$\int \rho(y)\,dy = 1\,. \tag{3.5.9}$$

Set

$$\psi(x) = \int \rho(y - x)\varphi(y)\,dy\,.$$

By Theorem 3.2.5 ψ is a $\mathcal{C}^\infty$ function. Moreover, if A_δ is the set of points in $\mathbb{R}^d$ whose distance in the sup norm from A is $\le \delta$ then for $x \notin A_\delta$ and $y \in A$, $|x - y| > \delta$ and hence $\rho(y - x) = 0$. Thus for $x \notin A_\delta$

$$\int \rho(y - x)\varphi(y)\,dy = \int_A \rho(y - x)\varphi(y)\,dy = 0\,,$$

so ψ is supported on the compact set A_δ. Moreover, since $\delta < \frac{d}{2}$, $\operatorname{supp}\psi$ is contained in V. Finally note that by (3.5.9) and exercise 4 of §3.4:

$$\int \rho(y - x)\,dy = \int \rho(y)\,dy = 1 \tag{3.5.10}$$

and hence

$$\varphi(x) \quad = \quad \int \varphi(x)\rho(y - x)\,dy$$

so

$$\varphi(x) - \psi(x) \quad = \quad \int (\varphi(x) - \varphi(y))\rho(y - x)\,dy$$

and

$$|\varphi(x) - \psi(x)| \;\; \leq \;\; \int |\varphi(x) - \varphi(y)|\, \rho(y-x)\, dy\,.$$

But $\rho(y-x) = 0$ for $|x-y| \geq \delta$; and $|\varphi(x)-\varphi(y)| < \epsilon$ for $|x-y| \leq \delta$, so the integrand on the right is less than

$$\epsilon \int \rho(y-x)\, dy\,,$$

and hence by (3.5.10)

$$|\varphi(x) - \psi(x)| \leq \epsilon\,.$$

□

To prove the identity (3.5.1), let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ cut-off function which is one on a neighborhood, V_1, of the support of φ, is non-negative, and is compactly supported with $\operatorname{supp}\gamma \subseteq V$, and let

$$c = \int \gamma(y)\, dy\,.$$

By Theorem 3.5.4 there exists, for every $\epsilon > 0$, a $\mathcal{C}^\infty$ function ψ, with support on V_1 satisfying

$$|\varphi - \psi| \leq \tfrac{\epsilon}{2c}\,. \tag{3.5.11}$$

Thus

$$\begin{aligned}\Big|\int_V (\varphi - \psi)(y)\, dy\Big| \;\; &\leq \;\; \int_V |\varphi - \psi|(y)\, dy \\ &\leq \;\; \int_V \gamma|\varphi - \psi|(xy)\, dy \\ &\leq \;\; \tfrac{\epsilon}{2c}\int \gamma(y)\, dy \leq \tfrac{\epsilon}{2}\end{aligned}$$

so

$$\Big|\int_V \varphi(y)\, dy - \int_V \psi(y)\, dy\Big| \leq \tfrac{\epsilon}{2}\,. \tag{3.5.12}$$

Similarly, the expression

$$\Big|\int_U (\varphi - \psi)\circ f(x) |\det Df(x)|\, dx\Big|$$

is less than or equal to the integral

$$\int_U \gamma \circ f(x) |(\varphi - \psi) \circ f(x)| \, |\det Df(x)| \, dx$$

and by (3.5.11), $|(\varphi - \psi) \circ f(x)| \leq \frac{\epsilon}{2c}$, so this integral is less than or equal to

$$\tfrac{\epsilon}{2c} \int \gamma \circ f(x) |\det Df(x)| \, dx$$

and hence by (3.5.1) is less than or equal to $\frac{\epsilon}{2}$. Thus
(3.5.13)

$$\left| \int_U \varphi \circ f(x) \, |\det Df(x)| dx - \int_U \psi \circ f(x) |\det Df(x)| \; dx \right| \leq \tfrac{\epsilon}{2} \, .$$

Combining (3.5.12), (3.5.13) and the identity

$$\int_V \psi(y) \, dy = \int \psi \circ f(x) |\det Df(x)| \, dx$$

we get, for all $\epsilon > 0$,

$$\left| \int_V \varphi(y) \, dy - \int_U \varphi \circ f(x) |\det Df(x)| \, dx \right| \leq \epsilon$$

and hence

$$\int \varphi(y) \, dy = \int \varphi \circ f(x) |\det Df(x)| \, dx \, .$$

Exercises for §3.5

1. Let $h : V \to \mathbb{R}$ be a non-negative continuous function. Show that if the improper integral

$$\int_V h(y) \, dy$$

is well-defined, then the improper integral

$$\int_U h \circ f(x) |\det Df(x)| \, dx$$

is well-defined and these two integrals are equal.

Hint: If φ_i, $i = 1, 2, 3, \dots$ is a partition of unity on V then $\psi_i = \varphi_i \circ f$ is a partition of unity on U and

$$\int \varphi_i h \, dy = \int \psi_i (h \circ f(x)) |\det Df(x)| \, dx \, .$$

Now sum both sides of this identity over i.

2. Show that the result above is true without the assumption that h is non-negative.

Hint: $h = h_+ - h_-$, where $h_+ = \max(h, 0)$ and $h_- = \max(-h, 0)$.

3. Show that, in the formula (3.4.2), one can allow the function, φ, to be a *continuous* compactly supported function rather than a $\mathcal{C}^\infty$ compactly supported function.

4. Let $\mathbb{H}^n$ be the half-space (**??**) and U and V open subsets of $\mathbb{R}^n$. Suppose $f : U \to V$ is an orientation preserving diffeomorphism mapping $U \cap \mathbb{H}^n$ onto $V \cap \mathbb{H}^n$. Show that for $\omega \in \Omega^n_c(V)$

$$\int_{U \cap \mathbb{H}^n} f^* \omega = \int_{V \cap \mathbb{H}^n} \omega \, . \tag{3.5.14}$$

Hint: Interpret the left and right hand sides of this formula as improper integrals over $U \cap \operatorname{Int} \mathbb{H}^n$ and $V \cap \operatorname{Int} \mathbb{H}^n$.

5. The boundary of $\mathbb{H}^n$ is the set

$$b\mathbb{H}^n = \{(0, x_2, \dots, x_n) \, , \quad (x_2, \dots, x_n) \in \mathbb{R}^n\}$$

so the map

$$\iota : \mathbb{R}^{n-1} \to \mathbb{H}^n \, , \quad (x_2, \dots, x_n) \to (0, x_2, \dots, x_n)$$

in exercise 9 in §3.2 maps $\mathbb{R}^{n-1}$ bijectively onto $b\mathbb{H}^n$.

(a) Show that the map $f : U \to V$ in exercise 4 maps $U \cap b\mathbb{H}^n$ onto $V \cap b\mathbb{H}^n$.

(b) Let $U' = \iota^{-1}(U)$ and $V' = \iota^{-1}(V)$. Conclude from part (a) that the restriction of f to $U \cap b\mathbb{H}^n$ gives one a diffeomorphism

$$g : U' \to V'$$

satisfying:

$$\iota \cdot g = f \cdot \iota \, . \tag{3.5.15}$$

(c) Let μ be in $\Omega_c^{n-1}(V)$. Conclude from (3.2.7) and (3.5.14):

$$\int_{U'} g^* \iota^* \mu = \int_{V'} \iota^* \mu \tag{3.5.16}$$

and in particular show that the diffeomorphism, $g : U' \to V'$, is orientation preserving.

3.6 Techniques for computing the degree of a mapping

Let U and V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ a proper $\mathcal{C}^\infty$ mapping. In this section we will show how to compute the degree of f and, in particular, show that it is always an integer. From this fact we will be able to conclude that the degree of f is a topological invariant of f: if we deform f smoothly, its degree doesn't change.

Definition 3.6.1. *A point, $x \in U$, is a* critical point *of f if the derivative*

$$Df(x) : \mathbb{R}^n \to \mathbb{R}^n$$

fails to be bijective, i.e., if $\det(Df(x)) = 0$.

We will denote the set of critical points of f by C_f. It's clear from the definition that this set is a closed subset of U and hence, by exercise 3 in §3.4, $f(C_f)$ is a closed subset of V. We will call this image the set of *critical values* of f and the complement of this image the set of *regular values* of f. Notice that $V - f(U)$ is contained in $f - f(C_f)$, so if a point, $g \in V$ is not in the image of f, it's a regular value of f "by default", i.e., it contains no points of U in the pre-image and hence, a fortiori, contains no critical points in its pre-image. Notice also that C_f can be quite large. For instance, if c is a point in V and $f : U \to V$ is the constant map which maps all of U onto c, then $C_f = U$. However, in this example, $f(C_f) = \{c\}$, so the set of regular values of f is $V - \{c\}$, and hence (in this example) is an open dense subset of V. We will show that this is true in general.

Theorem 3.6.2. *(Sard's theorem.)*

If U and V are open subsets of $\mathbb{R}^n$ and $f : U \to V$ a proper $\mathcal{C}^\infty$ map, the set of regular values of f is an open dense subset of V.

We will defer the proof of this to Section 3.7 and, in this section, explore some of its implications. Picking a regular value, q, of f we will prove:

Theorem 3.6.3. *The set, $f^{-1}(q)$ is a finite set. Moreover, if $f^{-1}(q) = \{p_1, \dots, p_n\}$ there exist connected open neighborhoods, U_i, of p_i in Y and an open neighborhood, W, of q in V such that:*

i. *for $i \neq j$ U_i and U_j are disjoint;*

ii. *$f^{-1}(W) = \bigcup U_i$,*

iii. *f maps U_i diffeomorphically onto W.*

Proof. If $p \in f^{-1}(q)$ then, since q is a regular value, $p \notin C_f$; so

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^n$$

is bijective. Hence by the inverse function theorem, f maps a neighborhood, U_p of p diffeomorphically onto a neighborhood of q. The open sets

$$\{U_p\,, \quad p \in f^{-1}(q)\}$$

are a covering of $f^{-1}(q)$; and, since f is proper, $f^{-1}(q)$ is compact; so we can extract a finite subcovering

$$\{U_{p_i}\,, \quad i = 1, \dots, N\}$$

and since p_i is the only point in U_{p_i} which maps onto q, $f^{-1}(q) = \{p_1, \dots, p_N\}$.

Without loss of generality we can assume that the U_{p_i}'s are disjoint from each other; for, if not, we can replace them by smaller neighborhoods of the p_i's which have this property. By Theorem 3.4.2 there exists a connected open neighborhood, W, of q in V for which

$$f^{-1}(W) \subset \bigcup U_{p_i}\,.$$

To conclude the proof let $U_i = f^{-1}(W) \cap U_{p_i}$.

□

The main result of this section is a recipe for computing the degree of f by counting the number of p_i's above, keeping track of orientation.

Theorem 3.6.4. *For each $p_i \in f^{-1}(q)$ let $\sigma_{p_i} = +1$ if $f : U_i \to W$ is orientation preserving and -1 if $f : U_i \to W$ is orientation reversing. Then*

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}\,. \tag{3.6.1}$$

Proof. Let ω be a compactly supported n-form on W whose integral is one. Then

$$\deg(f) = \int_U f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega\,.$$

Since $f : U_i \to W$ is a diffeomorphism

$$\int_{U_i} f^*\omega = \pm \int_W \omega = +1 \text{ or } -1$$

depending on whether $f : U_i \to W$ is orientation preserving or not. Thus $\deg(f)$ is equal to the sum (3.6.1).

□

As we pointed out above, a point, $q \in V$ can qualify as a regular value of f "by default", i.e., by not being in the image of f. In this case the recipe (3.6.1) for computing the degree gives "by default" the answer zero. Let's corroborate this directly.

Theorem 3.6.5. *If $f : U \to V$ isn't onto,* $\deg(f) = 0$.

Proof. By exercise 3 of §3.4, $V - f(U)$ is open; so if it is non-empty, there exists a compactly supported n-form, ω, with support in $V - f(U)$ and with integral equal to one. Since $\omega = 0$ on the image of f, $f^*\omega = 0$; so

$$0 = \int_U f^*\omega = \deg(f) \int_V \omega = \deg(f)\,.$$

□

Remark: In applications the contrapositive of this theorem is much more useful than the theorem itself.

Theorem 3.6.6. *If* $\deg(f) \neq 0$ *f maps U onto V.*

In other words if $\deg(f) \neq 0$ the equation

$$f(x) = y \tag{3.6.2}$$

has a solution, $x \in U$ for *every* $y \in V$.

We will now show that the degree of f is a topological invariant of f: if we deform f by a "homotopy" we don't change its degree. To make this assertion precise, let's recall what we mean by a *homotopy* between a pair of $\mathcal{C}^\infty$ maps. Let U be an open subset of $\mathbb{R}^m$, V an open subset of $\mathbb{R}^n$, A an open subinterval of $\mathbb{R}$ containing 0 and 1, and $f_i : U \to V$, $i = 0, 1$, $\mathcal{C}^\infty$ maps. Then a $\mathcal{C}^\infty$ map $F : U \times A \to V$ is a homotopy between f_0 and f_1 if $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. (See Definition **??**.) Suppose now that f_0 and f_1 are proper.

Definition 3.6.7. *F is a* proper *homotopy between f_0 and f_1 if the map*

$$F^\sharp : U \times A \to V \times A \tag{3.6.3}$$

mapping (x, t) to $(F(x, t), t)$ is proper.

Note that if F is a proper homotopy between f_0 and f_1, then for every t between 0 and 1, the map

$$f_t : U \to V\,, \quad f_t(x) = F_t(x)$$

is proper.

Now let U and V be open subsets of $\mathbb{R}^n$.

Theorem 3.6.8. *If f_0 and f_1 are properly homotopic, their degrees are the same.*

Proof. Let

$$\omega = \varphi(y)\, d\,y_1 \wedge \cdots \wedge d\,y_n$$

be a compactly supported n-form on X whose integral over V is 1. The the degree of f_t is equal to

$$\int_U \varphi(F_1(x, t), \ldots, F_n(x, t)) \det D_x F(x, t)\, dx\,. \tag{3.6.4}$$

The integrand in (3.6.4) is continuous and for $0 \le t \le 1$ is supported on a compact subset of $U \times [0, 1]$, hence (3.6.4) is continuous as a function of t. However, as we've just proved, $\deg(f_t)$ is *integer* valued so this function is a constant. □

(For an alternative proof of this result see exercise 9 below.) We'll conclude this account of degree theory by describing a couple applications.

Application 1. The Brouwer fixed point theorem

Let B^n be the closed unit ball in $\mathbb{R}^n$:

$$\{x \in \mathbb{R}^n \,,\ \|x\| \leq 1\}\,.$$

Theorem 3.6.9. *If $f : B^n \to B^n$ is a continuous mapping then f has a fixed point, i.e., maps some point, $x_0 \in B^n$ onto itself.*

The idea of the proof will be to assume that there isn't a fixed point and show that this leads to a contradiction. Suppose that for every point, $x \in B^n$ $f(x) \neq x$. Consider the ray through $f(x)$ in the direction of x:

$$f(x) + s(x - f(x))\,, \quad 0 \leq s < \infty\,.$$

This intersects the boundary, S^{n-1}, of B^n in a unique point, $\gamma(x)$, (see figure 1 below); and one of the exercises at the end of this section will be to show that the mapping $\gamma : B^n \to S^{n-1}$, $x \to \gamma(x)$, is a continuous mapping. Also it is clear from figure 1 that $\gamma(x) = x$ if $x \in S^{n-1}$, so we can extend γ to a continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ by letting γ be the identity for $\|x\| \geq 1$. Note that this extended mapping has the property

$$\|\gamma(x)\| \geq 1 \tag{3.6.5}$$

for all $x \in \mathbb{R}^n$ and

$$\gamma(x) = x \tag{3.6.6}$$

for all $\|x\| \geq 1$. To get a contradiction we'll show that γ can be approximated by a $\mathcal{C}^\infty$ map which has similar properties. For this we will need the following corollary of Theorem 3.5.4.

Lemma 3.6.10. *Let U be an open subset of $\mathbb{R}^n$, C a compact subset of U and $\varphi : U \to \mathbb{R}$ a continuous function which is $\mathcal{C}^\infty$ on the complement of C. Then for every $\epsilon > 0$, there exists a $\mathcal{C}^\infty$ function, $\psi : U \to \mathbb{R}$, such that $\varphi - \psi$ has compact support and $|\varphi - \psi| < \epsilon$.*

Proof. Let ρ be a bump function which is in $\mathcal{C}_0^\infty(U)$ and is equal to 1 on a neighborhood of C. By Theorem 3.5.4 there exists a function, $\psi_0 \in \mathcal{C}_0^\infty(U)$ such that $|\rho\varphi - \psi_0| < \epsilon$. Let $\psi = (1-\rho)\varphi + \psi_0$, and note that

$$\begin{aligned} \varphi - \psi &= (1-\rho)\varphi + \rho\varphi - (1-\rho)\varphi - \psi_0 \\ &= \rho\varphi - \psi_0\,. \end{aligned}$$

By applying this lemma to each of the coordinates of the map, γ, one obtains a $\mathcal{C}^\infty$ map, $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|g - \gamma\| < \epsilon < 1 \tag{3.6.7}$$

and such that $g = \gamma$ on the complement of a compact set. However, by (3.6.6), this means that g is equal to the identity on the complement of a compact set and hence (see exercise 9) that g is proper and has degree one. On the other hand by (3.6.8) and (3.6.6) $\|g(x)\| > 1-\epsilon$ for all $x \in \mathbb{R}^n$, so $0 \notin \operatorname{Im} g$ and hence by Theorem 3.6.4, $\deg(g) = 0$. Contradiction.

□

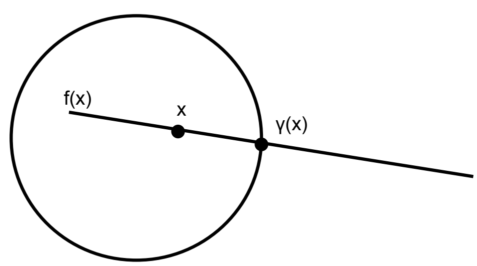


Figure 3.6.1.

Application 2. The fundamental theorem of algebra

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients. If we identify the complex plane

$$\mathbb{C} = \{z = x + iy\,;\, x, y \in \mathbb{R}\}$$

with $\mathbb{R}^2$ via the map, $(x, y) \in \mathbb{R}^2 \to z = x + iy$, we can think of p as defining a mapping

$$p : \mathbb{R}^2 \to \mathbb{R}^2\,,\; z \to p(z)\,.$$

We will prove

Theorem 3.6.11. *The mapping, p, is proper and* $\deg(p) = n$.

Proof. For $t \in \mathbb{R}$

$$\begin{aligned} p_t(z) &= (1-t)z^n + tp(z) \\ &= z^n + t\sum_{i=0}^{n-1} a_i z^i \,. \end{aligned}$$

We will show that the mapping

$$g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \,,\; z \to p_t(z)$$

is a proper homotopy. Let

$$C = \sup\{|a_i|\,,\, i = 0, \ldots, n-1\}\,.$$

Then for $|z| \geq 1$

$$\begin{aligned} |a_0 + \cdots + a_{n-1}z^{n-1}| &\leq |a_0| + |a_1||z| + \cdots + |a_{n-1}|\,|z|^{n-1} \\ &\leq C|z|^{n-1}\,, \end{aligned}$$

and hence, for $|t| \leq a$ and $|z| \geq 2aC$,

$$\begin{aligned} |p_t(z)| &\geq |z|^n - aC|z|^{n-1} \\ &\geq aC|z|^{n-1}\,. \end{aligned}$$

If A is a compact subset of $\mathbb{C}$ then for some $R > 0$, A is contained in the disk, $|w| \leq R$ and hence the set

$$\{z \in \mathbb{C}\,,\, (p_t(z), t) \in A \times [-a, a]\}$$

is contained in the compact set

$$\{z \in \mathbb{C}\,,\, aC|z|^{n-1} \leq R\}\,,$$

and this shows that g is a proper homotopy. Thus each of the mappings,

$$p_t : \mathbb{C} \to \mathbb{C}\,,$$

is proper and $\deg p_t = \deg p_1 = \deg p = \deg p_0$. However, $p_0 : \mathbb{C} \to \mathbb{C}$ is just the mapping, $z \to z^n$ and an elementary computation (see exercises 5 and 6 below) shows that the degree of this mapping is n. □

In particular for $n > 0$ the degree of p is non-zero; so by Theorem 3.6.4 we conclude that $p : \mathbb{C} \to \mathbb{C}$ is surjective and hence has zero in its image.

Theorem 3.6.12. *(fundamental theorem of algebra)*
Every polynomial,

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0 \,,$$

with complex coefficients has a complex root, $p(z_0) = 0$*, for some* $z_0 \in \mathbb{C}$*.*

Exercises for §3.6

1. Let W be a subset of $\mathbb{R}^n$ and let $a(x)$, $b(x)$ and $c(x)$ be real-valued functions on W of class C^r. Suppose that for every $x \in W$ the quadratic polynomial

$$(*) \qquad a(x)s^2 + b(x)s + c(x)$$

has two distinct real roots, $s_+(x)$ and $s_-(x)$, with $s_+(x) > s_-(x)$. Prove that s_+ and s_- are functions of class C^r.

Hint: What *are* the roots of the quadratic polynomial: $as^2 + bs + c$?

2. Show that the function, $\gamma(x)$, defined in figure 1 is a continuous mapping of B^n onto S^{2n-1}. *Hint:* $\gamma(x)$ lies on the ray,

$$f(x) + s(x - f(x)) \,, \quad 0 \le s < \infty$$

and satisfies $\|\gamma(x)\| = 1$; so $\gamma(x)$ is equal to

$$f(x) + s_0(x - f(x))$$

where s_0 is a non-negative root of the quadratic polynomial

$$\|f(x) + s(x - f(x))\|^2 - 1 \,.$$

Argue from figure 1 that this polynomial has to have two distinct real roots.

3. Show that the Brouwer fixed point theorem isn't true if one replaces the closed unit ball by the open unit ball. *Hint:* Let U be the open unit ball (i.e., the interior of B^n). Show that the map

$$h : U \to \mathbb{R}^n , \quad h(x) = \frac{x}{1 - \|x\|^2}$$

is a diffeomorphism of U onto $\mathbb{R}^n$, and show that there are lots of mappings of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which don't have fixed points.

4. Show that the fixed point in the Brouwer theorem doesn't have to be an interior point of B^n, i.e., show that it can lie on the boundary.

5. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the mapping: $(x, y) \to z = x + iy$, we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself, i.e., a mapping of the form

$$z \to cz , \quad c \in \mathbb{C}$$

as being an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself. Show that the determinant of this mapping is $|c|^2$.

6. (a) Let $f : \mathbb{C} \to \mathbb{C}$ be the mapping, $f(z) = z^n$. Show that

$$Df(z) = nz^{n-1} .$$

Hint: Argue from first principles. Show that for $h \in \mathbb{C} = \mathbb{R}^2$

$$\frac{(z + h)^n - z^n - nz^{n-1}h}{|h|}$$

tends to zero as $|h| \to 0$.

(b) Conclude from the previous exercise that

$$\det Df(z) = n^2|z|^{2n-2} .$$

(c) Show that at every point $z \in \mathbb{C} - 0$, f is orientation preserving.

(d) Show that every point, $w \in \mathbb{C} - 0$ is a regular value of f and that

$$f^{-1}(w) = \{z_1, \dots , z_n\}$$

with $\sigma_{z_i} = +1$.

(e) Conclude that the degree of f is n.

7. Prove that the map, f, in exercise 6 has degree n by deducing this directly from the definition of degree. *Some hints:*

(a) Show that in polar coordinates, f is the map, $(r, \theta) \to (r^n, n\theta)$.

(b) Let ω be the two-form, $g(x^2+y^2)\, dx \wedge dy$, where $g(t)$ is a compactly supported $\mathcal{C}^\infty$ function of t. Show that in polar coordinates, $\omega = g(r^2) r\, dr \wedge d\theta$, and compute the degree of f by computing the integrals of ω and $f^*\omega$, in polar coordinates and comparing them.

8. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, A an open subinterval of $\mathbb{R}$ containing 0 and 1, $f_i : U \to V$ $i = 0, 1$, a pair of $\mathcal{C}^\infty$ mappings and $F : U \times A \to V$ a homotopy between f_0 and f_1.

(a) In §2.3, exercise 4 you proved that if μ is in $\Omega^k(V)$ and $d\mu = 0$, then

$$f_0^*\mu - f_1^*\mu = d\nu \tag{3.6.8}$$

where ν is the $(k-1)$-form, $Q\alpha$, in formula (**??**). Show (by careful inspection of the definition of $Q\alpha$) that if F is a *proper* homotopy and $\mu \in \Omega_c^k(V)$ then $\nu \in \Omega_c^{k-1}(U)$.

(b) Suppose in particular that U and V are open subsets of $\mathbb{R}^n$ and μ is in $\Omega_c^n(V)$. Deduce from (3.6.8) that

$$\int f_0^*\mu = \int f_1^*\mu$$

and deduce directly from the definition of degree that degree is a proper homotopy invariant.

9. Let U be an open connected subset of $\mathbb{R}^n$ and $f : U \to U$ a proper $\mathcal{C}^\infty$ map. Prove that if f is equal to the identity on the complement of a compact set, C, then f is proper and its degree is equal to 1. *Hints:*

(a) Show that for every subset, A, of U, $f^{-1}(A) \subseteq A \cup C$, and conclude from this that f is proper.

(b) Let $C' = f(C)$. Use the recipe (1.6.1) to compute $\deg(f)$ with $q \in U - C'$.

10. Let $[a_{i,j}]$ be an $n \times n$ matrix and $A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with this matrix. Frobenius' theorem asserts: *If the $a_{i,j}$'s are non-negative then A has a non-negative eigenvalue.* In

other words there exists a $v \in \mathbb{R}^n$ and a $\lambda \in \mathbb{R}$, $\lambda \geq 0$, such that $Av = \lambda v$. Deduce this linear algebra result from the Brouwer fixed point theorem. *Hints:*

(a) We can assume that A is bijective, otherwise 0 is an eigenvalue. Let S^{n-1} be the $(n-1)$-sphere, $|x| = 1$, and $f : S^{n-1} \to S^{n-1}$ the map,

$$f(x) = \frac{Ax}{\|Ax\|}\,.$$

Show that f maps the set

$$Q = \{(x_1, \ldots, x_n) \in S^{n-1}\,; \quad x_i \geq 0\}$$

into itself.

(b) It's easy to prove that Q is homeomorphic to the unit ball B^{n-1}, i.e., that there exists a continuous map, $g : Q \to B^{n-1}$ which is invertible and has a continuous inverse. Without bothering to prove this fact deduce from it Frobenius' theorem.

3.7 Appendix: Sard's theorem

The version of Sard's theorem stated in §3.5 is a corollary of the following more general result.

Theorem 3.7.1. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a $\mathcal{C}^\infty$ map. Then $\mathbb{R}^n - f(C_f)$ is dense in $\mathbb{R}^n$.*

Before undertaking to prove this we will make a few general comments about this result.

Remark 3.7.2. *If $\mathcal{O}_n$, $n = 1, 2,$ are open dense subsets of $\mathbb{R}^n$, the intersection*

$$\bigcap_n \mathcal{O}_n$$

*is dense in $\mathbb{R}^n$. (See [**?**], pg. 200 or exercise 4 below.)*

Remark 3.7.3. *If A_n, $n = 1, 2, \ldots$ are a covering of U by compact sets, $\mathcal{O}_n = \mathbb{R}^n - f(C_f \cap A_n)$ is open, so if we can prove that it's dense then by Remark 3.7.2 we will have proved Sard's theorem. Hence since we can always cover U by a countable collection of closed cubes, it suffices to prove: for every closed cube, $A \subseteq U$, $\mathbb{R}^n - f(C_f \cap A)$ is dense in $\mathbb{R}^n$.*

Remark 3.7.4. *Let $g : W \to U$ be a diffeomorphism and let $h = f \circ g$. Then*

$$f(C_f) = h(C_h) \tag{3.7.1}$$

so Sard's theorem for g implies Sard's theorem for f.

We will first prove Sard's theorem for the set of *super-critical* points of f, the set:

$$C_f^\sharp = \{p \in U\,, \quad Df(p) = 0\}\,. \tag{3.7.2}$$

Proposition 3.7.5. *Let $A \subseteq U$ be a closed cube. Then the open set $\mathbb{R}^n - f(A \cap C_f^\sharp)$ is a dense subset of $\mathbb{R}^n$.*

We'll deduce this from the lemma below.

Lemma 3.7.6. *Given $\epsilon > 0$ one can cover $f(A \cap C_f^\sharp)$ by a finite number of cubes of total volume less than ϵ.*

Proof. Let the length of each of the sides of A be ℓ. Given $\delta > 0$ one can subdivide A into N^n cubes, each of volume, $\left(\frac{\ell}{N}\right)^n$, such that if x and y are points of any one of these subcubes

$$\left|\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y)\right| < \delta\,. \tag{3.7.3}$$

Let $A_1, \dots, A_m$ be the cubes in this collection which intersect $C_f^\sharp$. Then for $z_0 \in A_i \cap C_f^\sharp$, $\frac{\partial f_i}{\partial x_j}(z_0) = 0$, so for $z \in A_i$

$$\left|\frac{\partial f_i}{\partial x_j}(z)\right| < \delta \tag{3.7.4}$$

by (3.7.3). If x and y are points of A_i then by the mean value theorem there exists a point z on the line segment joining x to y such that

$$f_i(x) - f_i(y) = \sum \frac{\partial f_i}{\partial x_j}(z)(x_j - y_j)$$

and hence by (3.7.4)

$$|f_i(x) - f_i(y)| \le \delta \sum |x_i - y_i| \le n\delta \frac{\ell}{N}\,. \tag{3.7.5}$$

Thus $f(C_f \cap A_i)$ is contained in a cube, B_i, of volume $\left(n\frac{\delta\ell}{N}\right)^n$, and $f(C_f \cap A)$ is contained in a union of cubes, B_i, of total volume less that

$$N^n n^n \frac{\delta^n \ell^n}{N^n} = n^n \delta^n \ell^n$$

so if w choose $\delta^n \ell^n < \epsilon$, we're done.

□

Proof. To prove Proposition 3.7.5 we have to show that for every point $p \in \mathbb{R}^n$ and neighborhood, W, of p, $W - f(C_f^\sharp \cap A)$ is non-empty. Suppose

$$W \subseteq f(C_f^\sharp \cap A)\,. \tag{3.7.6}$$

Without loss of generality we can assume W is a cube of volume ϵ, but the lemma tells us that $f(C_f^\sharp \cap A)$ can be covered by a finite number of cubes whose total volume is *less* than ϵ, and hence by (3.7.6) W can be covered by a finite number of cubes of total volume less than ϵ, so its volume is less than ϵ. This contradiction proves that the inclusion (3.7.6) can't hold.

□

To prove Theorem 3.7.1 let $U_{i,j}$ be the subset of U where $\frac{\partial f_i}{\partial x_j} \neq 0$. Then

$$U = \bigcup U_{i,j} \cup C_f^\sharp\,,$$

so to prove the theorem it suffices to show that $\mathbb{R}^n - f(U_{i,j} \cap C_f)$ is dense in $\mathbb{R}^n$, i.e., it suffices to prove the theorem with U replaced by $U_{i,j}$. Let $\sigma_i : \mathbb{R}^n \times \mathbb{R}^n$ be the involution which interchanges x_1 and x_i and leaves the remaining x_k's fixed. Letting $f_{\text{new}} = \sigma_i f_{\text{old}} \sigma_j$ and $U_{\text{new}} = \sigma_j U_{\text{old}}$, we have, for $f = f_{\text{new}}$ and $U = U_{\text{new}}$

$$\frac{\partial f_1}{\partial x_1}(p) \neq 0 \quad \text{for all } p \in U\} \tag{3.7.7}$$

so we're reduced to proving Theorem 3.7.1 for maps $f : U \to \mathbb{R}^n$ having the property (3.7.6). Let $g : U \to \mathbb{R}^n$ be defined by

$$g(x_1, \ldots, x_n) = (f_1(x), x_2, \ldots, x_n)\,. \tag{3.7.8}$$

Then

$$g^*x_1 = f^*x_1 \quad = \quad f_1(x_1, \ldots, x_n) \tag{3.7.9}$$

and

$$\det(Dg) \quad = \quad \frac{\partial f_1}{\partial x_1} \neq 0\,. \tag{3.7.10}$$

Thus, by the inverse function theorem, g is locally a diffeomorphism at every point, $p \in U$. This means that if A is a compact subset of U we can cover A by a finite number of open subsets, $U_i \subset U$ such that g maps U_i diffeomorphically onto an open subset W_i in $\mathbb{R}^n$. To conclude the proof of the theorem we'll show that $\mathbb{R}^n - f(C_f \cap U_i \cap A)$ is a dense subset of $\mathbb{R}^n$. Let $h : W_i \to \mathbb{R}^n$ be the map $h = f \circ g^{-1}$. To prove this assertion it suffices by Remark 3.7.4 to prove that the set

$$\mathbb{R}^n - h(C_h)$$

is dense in $\mathbb{R}^n$. This we will do by induction on n. First note that for $n = 1$, $C_f = C_f^\sharp$, so we've already proved Theorem 3.7.1 in dimension one. Now note that by (3.7.8), $h^*x_1 = x_1$, i.e., h is a mapping of the form

$$h(x_1, \ldots, x_n) = (x_1, h_2(x), \ldots, h_n(x))\,. \tag{3.7.11}$$

Thus if we let W_c be the set

$$\{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\,;\, (c, x_2, \ldots, x_n) \in W_i\} \tag{3.7.12}$$

and let $h_c : W_c \to \mathbb{R}^{n-1}$ be the map

$$h_c(x_2, \ldots, x_n) = (h_2(c, x_2, \ldots, x_n), \ldots, h_n(c, x_2, \ldots, x_n))\,. \tag{3.7.13}$$

Then

$$\det(Dh_c)(x_2, \ldots, x_n) = \det(Dh)(c, x_2, \ldots, x_n) \tag{3.7.14}$$

and hence

$$(c, x) \in W_i \cap C_h \Leftrightarrow x \in C_{h_c}\,. \tag{3.7.15}$$

Now let $p_0 = (c, x_0)$ be a point in $\mathbb{R}^n$. We have to show that every neighborhood, V, of p_0 contains a point $p \in \mathbb{R}^n - h(C_h)$. Let $V_c \subseteq \mathbb{R}^{n-1}$ be the set of points, x, for which $(c, x) \in V$. By induction V_c contains a point, $x \in \mathbb{R}^{n-1} - h_c(C_{h_c})$ and hence $p = (c, x)$ is in V by definition and in $\mathbb{R}^n - h(C_n)$ by (3.7.15).

Q.E.D.

Exercises for §3.7

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be the map $f(x) = (x^2 - 1)^2$. What is the set of critical points of f? What is its image?

(b) Same questions for the map $f(x) = \sin x + x$.

(c) Same questions for the map

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases} .$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map, i.e., a map of the form

$$f(x) = A(x) + x_0$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. Prove Sard's theorem for f.

3. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which is supported in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and has a maximum at the origin. Let $r_1, r_2, \ldots,$ be an enumeration of the rational numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be the map

$$f(x) = \sum_{i=1}^{\infty} r_i \rho(x - i) .$$

Show that f is a $\mathcal{C}^\infty$ map and show that the image of C_f is dense in $\mathbb{R}$. (The moral of this example: Sard's theorem says that the complement of C_f is dense in $\mathbb{R}$, but C_f can be dense as well.)

4. Prove the assertion made in Remark 3.7.2. *Hint:* You need to show that for every point $p \in \mathbb{R}^n$ and every neighborhood, V, of p, $\bigcap \mathcal{O}_n \cap V$ is non-empty. Construct, by induction, a family of closed balls, B_k, such that

(a) $B_k \subseteq V$

(b) $B_{k+1} \subseteq B_k$

(c) $B_k \subseteq \bigcap\limits_{n \leq k} \mathcal{O}_n$

(d) radius $B_k < \frac{1}{k}$

and show that the intersection of the B_k's is non-empty.

5. Verify (3.7.1).

CHAPTER 4

FORMS ON MANIFOLDS

4.1 Manifolds

Our agenda in this chapter is to extend to manifolds the results of Chapters 2 and 3 and to formulate and prove manifold versions of two of the fundamental theorems of integral calculus: Stokes' theorem and the divergence theorem. In this section we'll define what we mean by the term "manifold", however, before we do so, a word of encouragement. Having had a course in multivariable calculus, you are already familiar with manifolds, at least in their one and two dimensional emanations, as curves and surfaces in $\mathbb{R}^3$, i.e., a manifold is basically just an n-dimensional surface in some high dimensional Euclidean space. To make this definition precise let X be a subset of $\mathbb{R}^N$, Y a subset of $\mathbb{R}^n$ and $f : X \to Y$ a continuous map. We recall

Definition 4.1.1. *f is a $\mathcal{C}^\infty$ map if for every $p \in X$, there exists a neighborhood, U_p, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $g_p : U_p \to \mathbb{R}^n$, which coincides with f on $U_p \cap X$.*

We also recall:

Theorem 4.1.2. *If $f : X \to Y$ is a $\mathcal{C}^\infty$ map, there exists a neighborhood, U, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $g : U \to \mathbb{R}^n$ such that g coincides with f on X.*

(A proof of this can be found in Appendix A.)

We will say that f is a *diffeomorphism* if it is one–one and onto and f and f^{-1} are both $\mathcal{C}^\infty$ maps. In particular if Y is an open subset of $\mathbb{R}^n$, X is an example of an object which we will call a *manifold*. More generally,

Definition 4.1.3. *A subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold if, for every $p \in X$, there exists a neighborhood, V, of p in $\mathbb{R}^m$, an open subset, U, in $\mathbb{R}^n$, and a diffeomorphism $\varphi : U \to X \cap V$.*

Thus X is an n-dimensional manifold if, locally near every point p, X "looks like" an open subset of $\mathbb{R}^n$.

Some examples:

1. *Graphs of functions.* Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^\infty$ function. Its graph

$$\Gamma_f = \{(x,t) \in \mathbb{R}^{n+1} ; \quad x \in U \, , \, t = f(x)\}$$

is an n-dimensional manifold in $\mathbb{R}^{n+1}$. In fact the map

$$\varphi : U \to \mathbb{R}^{n+1} \, , \quad x \to (x, f(x))$$

is a diffeomorphism of U onto Γ_f. (It's clear that φ is a $\mathcal{C}^\infty$ map, and it is a diffeomorphism since its inverse is the map, $\pi : \Gamma_f \to U$, $\pi(x,t) = x$, which is also clearly $\mathcal{C}^\infty$.)

2. *Graphs of mappings.* More generally if $f : U \to \mathbb{R}^k$ is a $\mathcal{C}^\infty$ map, its graph

$$\Gamma_f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \, , \quad x \in U \, , \, y = f(x)\}$$

is an n-dimensional manifold in $\mathbb{R}^{n+k}$.

3. *Vector spaces.* Let V be an n- dimensional vector subspace of $\mathbb{R}^N$, and $(e_1, \ldots, e_n)$ a basis of V. Then the linear map

$$\varphi : \mathbb{R}^n \to V \, , \quad (x_1, \ldots, x_n) \to \sum x_i e_i \tag{4.1.1}$$

is a diffeomorphism of $\mathbb{R}^n$ onto V. Hence every n-dimensional vector subspace of $\mathbb{R}^N$ is automatically an n-dimensional submanifold of $\mathbb{R}^N$. Note, by the way, that if V is *any* n-dimensional vector space, not necessarily a subspace of $\mathbb{R}^N$, the map (4.1.1) gives us an identification of V with $\mathbb{R}^n$. This means that we can speak of subsets of V as being k-dimensional submanifolds if, via this identification, they get mapped onto k-dimensional submanifolds of $\mathbb{R}^n$. (This is a trivial, but useful, observation since a lot of interesting manifolds occur "in nature" as subsets of some abstract vector space rather than explicitly as subsets of some $\mathbb{R}^n$. An example is the manifold, $O(n)$, of orthogonal $n \times n$ matrices. (See example 10 below.) This manifold occurs in nature as a submanifold of the vector space of n by n matrices.)

4. *Affine subspaces* of $\mathbb{R}^n$. These are manifolds of the form $p + V$, where V is a vector subspace of $\mathbb{R}^N$, and p is some specified point in

$\mathbb{R}^N$. In other words, they are diffeomorphic copies of the manifolds in example 3 with respect to the diffeomorphism

$$\tau_p : \mathbb{R}^N \times \mathbb{R}^N , \quad x \to x + p .$$

If X is an arbitrary submanifold of $\mathbb{R}^N$ its *tangent space* a point, $p \in X$, is an example of a manifold of this type. (We'll have more to say about tangent spaces in §4.2.)

5. *Product manifolds.* Let X_i, $i = 1, 2$ be an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$. Then the Cartesian product of X_1 and X_2

$$X_1 \times X_2 = \{(x_1, x_2) ; x_i \in X_i\}$$

is an n-dimensional submanifold of $\mathbb{R}^N$ where $n = n_1 + n_2$ and $\mathbb{R}^N = \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$.

We will leave for you to verify this fact as an exercise. *Hint:* For $p_i \in X_i$, $i = 1, 2$, there exists a neighborhood, V_i, of p_i in $\mathbb{R}^{N_i}$, an open set, U_i in $\mathbb{R}^{n_i}$, and a diffeomorphism $\varphi : U_i \to X_i \cap V_i$. Let $U = U_1 \times U_2$, $V = V_1 \times V_2$ and $X = X_1 \times X_2$, and let $\varphi : U \to X \cap V$ be the product diffeomorphism, $(\varphi(q_1), \varphi_2(q_2))$.

6. *The unit n-sphere.* This is the set of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{x \in \mathbb{R}^{n+1} , \quad x_1^2 + \cdots + x_{n+1}^2 = 1\} .$$

To show that S^n is an n-dimensional manifold, let V be the open subset of $\mathbb{R}^{n+1}$ on which x_{n+1} is positive. If U is the open unit ball in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is the function, $f(x) = (1 - (x_1^2 + \cdots + x_n^2))^{1/2}$, then $S^n \cap V$ is just the graph, Γ_f, of f as in example 1. So, just as in example 1, one has a diffeomorphism

$$\varphi : U \to S^n \cap V .$$

More generally, if $p = (x_1, \ldots, x_{n+1})$ is any point on the unit sphere, then x_i is non-zero for some i. If x_i is positive, then letting σ be the transposition, $i \leftrightarrow n+1$ and $f_\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, the map

$$f_\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

one gets a diffeomorphism, $f_\sigma \circ \varphi$, of U onto a neighborhood of p in S^n and if x_i is negative one gets such a diffeomorphism by replacing f_σ by $-f_\sigma$. In either case we've shown that for every point, p, in S^n, there is a neighborhood of p in S^n which is diffeomorphic to U.

7. *The 2-torus.* In calculus books this is usually described as the surface of rotation in $\mathbb{R}^3$ obtained by taking the unit circle centered at the point, $(2,0)$, in the (x_1, x_3) plane and rotating it about the x_3-axis. However, a slightly nicer description of it is as the product manifold $S^1 \times S^1$ in $\mathbb{R}^4$. (*Exercise:* Reconcile these two descriptions.)

We'll now turn to an alternative way of looking at manifolds: as *solutions of systems of equations*. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map.

Definition 4.1.4. *A point, $a \in \mathbb{R}^k$, is a regular value of f if for every point, $p \in f^{-1}(a)$, f is a submersion at p.*

Note that for f to be a submersion at p, $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ has to be onto, and hence k has to be less than or equal to N. Therefore this notion of "regular value" is interesting only if $N \geq k$.

Theorem 4.1.5. *Let $N - k = n$. If a is a regular value of f, the set, $X = f^{-1}(a)$, is an n-dimensional manifold.*

Proof. Replacing f by $\tau_{-a} \circ f$ we can assume without loss of generality that $a = 0$. Let $p \in f^{-1}(0)$. Since f is a submersion at p, the canonical submersion theorem (see Appendix B, Theorem 2) tells us that there exists a neighborhood, $\mathcal{O}$, of 0 in $\mathbb{R}^N$, a neighborhood, U_0, of p in U and a diffeomorphism, $g : \mathcal{O} \to U_0$ such that

$$f \circ g = \pi \tag{4.1.2}$$

where π is the projection map

$$\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k, \quad (x,y) \to x\,.$$

Hence $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n = \mathbb{R}^n$ and by (4.1.1), g maps $\mathcal{O} \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. However, $\mathcal{O} \cap \pi^{-1}(0)$ is a neighborhood, V, of 0 in $\mathbb{R}^n$ and $U_0 \cap f^{-1}(0)$ is a neighborhood of p in X, and, as remarked, these two neighborhoods are diffeomorphic. □

Some examples:

8. *The n-sphere.* Let

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

be the map,

$$(x_1, \dots, x_{n+1}) \to x_1^2 + \cdots + x_{n+1}^2 - 1\,.$$

Then

$$Df(x) = 2(x_1, \dots, x_{n+1})$$

so, if $x \neq 0$ f is a submersion at x. In particular f is a submersion at all points, x, on the n-sphere

$$S^n = f^{-1}(0)$$

so the n-sphere is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.

9. *Graphs.* Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and as in example 2 let

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k\,, \quad y = g(x)\}\,.$$

We claim that Γ_f is an n-dimensional submanifold of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$.

Proof. Let

$$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$$

be the map, $f(x, y) = y - g(x)$. Then

$$Df(x, y) = [-Dg(x)\,,\, I_k]$$

where I_k is the identity map of $\mathbb{R}^k$ onto itself. This map is always of rank k. Hence $\Gamma_f = f^{-1}(0)$ is an n-dimensional submanifold of R^{n+k}.

□

10. Let $\mathcal{M}_n$ be the set of all $n \times n$ matrices and let $\mathcal{S}_n$ be the set of all symmetric $n \times n$ matrices, i.e., the set

$$\mathcal{S}_n = \{A \in \mathcal{M}_n\,,\, A = A^t\}\,.$$

The map

$$[a_{i,j}] \to (a_{11}, a_{12}, \dots, a_{1n}, a_{2,1}, \dots, a_{2n}, \dots)$$

gives us an identification

$$\mathcal{M}_n \cong \mathbb{R}^{n^2}$$

and the map

$$[a_{i,j}] \to (a_{11}, \dots a_{1n}, a_{22}, \dots a_{2n}, a_{33}, \dots a_{3n}, \dots)$$

gives us an identification

$$\mathcal{S}_n \cong \mathbb{R}^{\frac{n(n+1)}{2}} .$$

(Note that if A is a symmetric matrix,

$$a_{12} = a_{21} \, , \; a_{13} = a_{31} \, , \; a_{32} = a_{23}, \text{ etc.}$$

so this map avoids redundancies.) Let

$$O(n) = \{A \in \mathcal{M}_n \, , \; A^t A = I\} \, .$$

This is the set of *orthogonal* $n \times n$ matrices, and we will leave for you as an exercise to show that it's an $n(n-1)/2$-dimensional manifold.

Hint: Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map $f(A) = A^t A - I$. Then

$$O(n) = f^{-1}(0) \, .$$

These examples show that lots of interesting manifolds arise as zero sets of submersions, $f : U \to \mathbb{R}^k$. This is, in fact, not just an accident. We will show that locally *every* manifold arises this way. More explicitly let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold, p a point of X, U a neighborhood of 0 in $\mathbb{R}^n$, V a neighborhood of p in $\mathbb{R}^N$ and $\varphi : (U, 0) \to (V \cap X, p)$ a diffeomorphism. We will for the moment think of φ as a $\mathcal{C}^\infty$ map $\varphi : U \to \mathbb{R}^N$ whose image happens to lie in X.

Lemma 4.1.6. *The linear map*

$$D\varphi(0) : \mathbb{R}^n \to \mathbb{R}^N$$

is injective.

Proof. $\varphi^{-1} : V \cap X \to U$ is a diffeomorphism, so, shrinking V if necessary, we can assume that there exists a $\mathcal{C}^\infty$ map $\psi : V \to U$ which coincides with φ^{-1}on $V \cap X$ Since φ maps U onto $V \cap X$, $\psi \circ \varphi = \varphi^{-1} \circ \varphi$ is the identity map on U. Therefore,

$$D(\psi \circ \varphi)(0) = (D\psi)(p) D\varphi(0) = I$$

by the chain rule, and hence if $D\varphi(0)v = 0$, it follows from this identity that $v = 0$.

□

Lemma 4.1.6 says that φ is an immersion at 0, so by the canonical immersion theorem (see Appendix B,Theorem 4) there exists a neighborhood, U_0, of 0 in U, a neighborhood, V_p, of p in V, and a diffeomorphism

$$g : (V_p, p) \to (U_0 \times \mathbb{R}^{N-n}, 0) \tag{4.1.3}$$

such that

$$g \circ \varphi = \iota \,, \tag{4.1.4}$$

ι being, as in Appendix B, the canonical immersion

$$\iota : U_0 \to U_0 \times \mathbb{R}^{N-n} \,, \quad x \to (x, 0) \,. \tag{4.1.5}$$

By (4.1.3) g maps $\varphi(U_0)$ diffeomorphically onto $\iota(U_0)$. However, by (4.1.2) and (4.1.3) $\iota(U_0)$ is defined by the equations, $x_i = 0$, $i = n+1, \dots, N$. Hence if $g = (g_1, \dots, g_N)$ the set, $\varphi(U_0) = V_p \cap X$ is defined by the equations

$$g_i = 0 \,, \quad i = n+1, \dots, N \,. \tag{4.1.6}$$

Let $\ell = N - n$, let

$$\pi : \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^\ell$$

be the canonical submersion,

$$\pi(x_1, \dots, x_N) = (x_{n+1}, \dots x_N)$$

and let $f = \pi \circ g$. Since g is a diffeomorphism, f is a submersion and (4.1.5) can be interpreted as saying that

$$V_p \cap X = f^{-1}(0) \,. \tag{4.1.7}$$

Thus to summarize we've proved

Theorem 4.1.7. *Let X be an n-dimensional submanifold of $\mathbb{R}^N$ and let $\ell = N - n$. Then for every $p \in X$ there exists a neighborhood, V_p, of p in $\mathbb{R}^N$ and a submersion*

$$f : (V_p, p) \to (\mathbb{R}^\ell, 0)$$

such that $X \cap V_p$ is defined by the equation (4.1.6).

A nice way of thinking about Theorem 4.1.2 is in terms of the coordinates of the mapping, f. More specifically if $f = (f_1, \dots, f_k)$ we can think of $f^{-1}(a)$ as being the set of solutions of the system of equations

$$f_i(x) = a_i\,, \quad i = 1, \dots, k \tag{4.1.8}$$

and the condition that a be a regular value of f can be interpreted as saying that for every solution, p, of this system of equations the vectors

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(0)\, dx_j \tag{4.1.9}$$

in $T_p^*\mathbb{R}^n$ are linearly independent, i.e., the system (4.1.7) is an "independent system of defining equations" for X.

Exercises.

1. Show that the set of solutions of the system of equations

$$x_1^2 + \cdots + x_n^2 \quad = \quad 1$$

and

$$x_1 + \cdots + x_n \quad = \quad 0$$

is an $n-2$-dimensional submanifold of $\mathbb{R}^n$.

2. Let S^{n-1} be the n-sphere in $\mathbb{R}^n$ and let

$$X_a = \{x \in S^{n-1}\,, \quad x_1 + \cdots + x_n = a\}\,.$$

For what values of a is X_a an $(n-2)$-dimensional submanifold of S^{n-1}?

3. Show that if X_i, $i = 1, 2$, is an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$ then

$$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

is an $(n_1 + n_2)$-dimensional submanifold of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

4. Show that the set

$$X = \{(x, \mathrm{v}) \in S^{n-1} \times \mathbb{R}^n, \quad x \cdot \mathrm{v} = 0\}$$

is a $2n-2$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. (Here "$x \cdot \mathrm{v}$" is the dot product, $\sum x_i v_i$.)

5. Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let $X = \operatorname{graph} g$. Prove directly that X is an n-dimensional manifold by proving that the map

$$\gamma : \mathbb{R}^n \to X, \qquad x \to (x, g(x))$$

is a diffeomorphism.

6. Prove that $O(n)$ is an $n(n-1)/2$-dimensional manifold. *Hints:*

(a) Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map

$$f(A) = A^t A = I .$$

Show that $O(n) = f^{-1}(0)$.

(b) Show that

$$f(A + \epsilon B) = A^t A + \epsilon(A^t B + B^t A) + \epsilon^2 B^t B .$$

(c) Conclude that the derivative of f at A is the map

$$B \in \mathcal{M}_n \to A^t B + B^t A . \tag{*}$$

(d) Let A be in $O(n)$. Show that if C is in $\mathcal{S}_n$ and $B = AC/2$ then the map, (*), maps B onto C.

(e) Conclude that the derivative of f is surjective at A.

(f) Conclude that 0 is a regular value of the mapping, f.

7. The next five exercises, which are somewhat more demanding than the exercises above, are an introduction to "Grassmannian" geometry.

(a) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and let $W = \operatorname{span}\{e_{k+1}, \ldots, e_n\}$. Prove that if V is a k-dimensional subspace of $\mathbb{R}^n$ and

$$V \cap W = \{0\} , \tag{1.1}$$

then one can find a *unique* basis of V of the form

$$\mathrm{v}_i = e_i + \sum_{j=1}^{\ell} b_{i,j} e_{k+j}\,, \quad i = 1, \dots, k\,, \tag{1.2}$$

where $\ell = n - k$.

(b) Let G_k be the set of k-dimensional subspaces of $\mathbb{R}^n$ having the property (1.1) and let $\mathcal{M}_{k,\ell}$ be the vector space of $k \times \ell$ matrices. Show that one gets from the identities (1.2) a bijective map:

$$\gamma : \mathcal{M}_{k,\ell} \to G_k\,. \tag{1.3}$$

8. Let S_n be the vector space of linear mappings of $\mathbb{R}^n$ into itself which are self-adjoint, i.e., have the property $A = A^t$.

(a) Given a k-dimensional subspace, V of $\mathbb{R}^n$ let $\pi_V : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection of $\mathbb{R}^n$ onto V. Show that π_V is in S_n and is of rank k, and show that $(\pi_V)^2 = \pi_V$.

(b) Conversely suppose A is an element of S_n which is of rank k and has the property, $A^2 = A$. Show that if V is the image of A in $\mathbb{R}^n$, then $A = \pi_V$.

Notation. We will call an $A \in S_n$ of the form, $A = \pi_V$ above a *rank k projection operator.*

9. Composing the map

$$\rho : G_k \to S_n\,, \quad V \to \pi_V \tag{1.4}$$

with the map (1.3) we get a map

$$\varphi : \mathcal{M}_{k,\ell} \to S_n\,, \quad \varphi = \rho \cdot \gamma\,. \tag{1.5}$$

Prove that φ is $\mathcal{C}^\infty$.

Hints:

(a) By Gram–Schmidt one can convert (1.2) into an orthonormal basis

$$e_{1,B}, \dots, e_{n,B} \tag{1.6}$$

of V. Show that the $e_{i,B}$'s are $\mathcal{C}^\infty$ functions of the matrix, $B = [b_{i,j}]$.

(b) Show that π_V is the linear mapping

$$\mathrm{v} \in V \to \sum_{i=1}^{k} (\mathrm{v} \cdot e_{i,B}) e_{i,B} \,.$$

10. Let $V_0 = \operatorname{span}\{e_1, \dots, e_n\}$ and let $\widetilde{G}_k = \rho(G_k)$. Show that φ maps a neighborhood of 0 in $\mathcal{M}_{k,\ell}$ diffeomorphically onto a neighborhood of π_{V_0} in $\widetilde{G}_k$.

Hints: π_V is in $\widetilde{G}_k$ if and only if V satisfies (1.1). For $1 \le i \le k$ let

$$w_i = \pi_V(e_i) = \sum_{j=1}^{k} a_{i,j} e_j + \sum_{r=1}^{\ell} c_{i,r} e_{k+r} \,. \tag{1.7}$$

(a) Show that if the matrix $A = [a_{i,j}]$ is invertible, π_V is in $\widetilde{G}_k$.

(b) Let $\mathcal{O} \subseteq \widetilde{G}_k$ be the set of all π_V's for which A is invertible. Show that $\varphi^{-1} : \mathcal{O} \to \mathcal{M}_{k,\ell}$ is the map

$$\varphi^{-1}(\pi_V) = B = A^{-1}C$$

where $C = [c_{i,j}]$.

11. Let $G(k,n) \subseteq S_n$ be the set of rank k projection operators. Prove that $G(k,n)$ is a $k\ell$-dimensional submanifold of the Euclidean space, $S_n = \mathbb{R}^{\frac{n(n+1)}{2}}$.

Hints:

(a) Show that if V is any k-dimensional subspace of $\mathbb{R}^n$ there exists a linear mapping, $A \in O(n)$ mapping V_0 to V.

(b) Show that $\pi_V = A \pi_{V_0} A^{-1}$.

(c) Let $K_A : S_n \to S_n$ be the linear mapping,

$$K_A(B) = ABA^{-1} \,.$$

Show that

$$K_A \cdot \varphi : \mathcal{M}_{k,\ell} \to S_n$$

maps a neighborhood of 0 in $\mathcal{M}_{k,\ell}$ diffeomorphically onto a neighborhood of π_V in $G(k,n)$.

Remark 4.1.8. *Let $Gr(k,n)$ be the set of all k-dimensional subspaces of $\mathbb{R}^n$. The identification of $Gr(k,n)$ with $G(k,n)$ given by $V \leftrightarrow \pi_V$ allows us to restate the result above in the form.*

The "Grassmannnian" Theorem: The set $(\mathrm{Gr}(k,n))$ (a.k.a. the "Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$") is a $k\ell$-dimensional submanifold of $S_n = \mathbb{R}^{\frac{n(n+1)}{2}}$.

12. Show that $\mathrm{Gr}(k,n)$ is a compact submanifold of S_n. *Hint:* Show that it's closed and bounded.

4.2 Tangent spaces

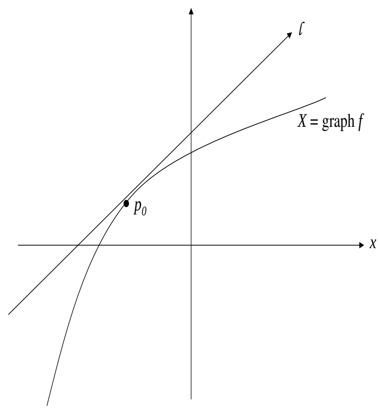
We recall that a subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold, if, for every $p \in X$, there exists an open set, $U \subseteq \mathbb{R}^n$, a neighborhood, V, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$-diffeomorphism, $\varphi : U \to X \cap X$.

Definition 4.2.1. *We will call φ a* parametrization *of X at p.*

Our goal in this section is to define the notion of the *tangent space*, T_pX, to X at p and describe some of its properties. Before giving our official definition we'll discuss some simple examples.

Example 1.

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $X = \mathrm{graph} f$.



Then in this figure above the tangent line, ℓ, to X at $p_0 = (x_0, y_0)$ is defined by the equation

$$y - y_0 = a(x - x_0)$$

where $a = f'(x_0)$ In other words if p is a point on ℓ then $p = p_0 + \lambda \mathrm{v}_0$ where $\mathrm{v}_0 = (1, a)$ and $\lambda \in \mathbb{R}$. We would, however, like the tangent space to X at p_0 to be a subspace of the tangent space to $\mathbb{R}^2$ at p_0, i.e., to be the subspace of the space: $T_{p_0}\mathbb{R}^2 = \{p_0\} \times \mathbb{R}^2$, and this we'll achieve by defining

$$T_{p_0}X = \{(p_0, \lambda \mathrm{v}_0)\,, \quad \lambda \in \mathbb{R}\}\,.$$

Example 2.

Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. The tangent plane to S^2 at p_0 is usually defined to be the plane

$$\{p_0 + \mathrm{v}\,;\, v \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}\,.$$

However, this tangent plane is easily converted into a subspace of $T_p\mathbb{R}^3$ via the map, $p_0 + \mathrm{v} \to (p_0, \mathrm{v})$ and the image of this map

$$\{(p_0, \mathrm{v})\,;\, \mathrm{v} \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}$$

will be our definition of $T_{p_0}S^2$.

Let's now turn to the general definition. As above let X be an n-dimensional submanifold of $\mathbb{R}^N$, p a point of X, V a neighborhood of p in $\mathbb{R}^N$, U an open set in $\mathbb{R}^n$ and

$$\varphi : (U, q) \to (X \cap V, p)$$

a parameterization of X. We can think of φ as a $\mathcal{C}^\infty$ map

$$\varphi : (U, q) \to (V, p)$$

whose image happens to lie in $X \cap V$ and we proved in §4.1 that its derivative at q

$$(d\varphi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N \tag{4.2.1}$$

is injective.

Definition 4.2.2. *The tangent space, T_pX, to X at p is the image of the linear map (4.2.1). In other words, $w \in T_p\mathbb{R}^N$ is in T_pX if and only if $w = d\varphi_q(\mathrm{v})$ for some $\mathrm{v} \in T_q\mathbb{R}^n$. More succinctly,*

$$T_pX = (d\varphi_q)(T_q\mathbb{R}^n)\,. \tag{4.2.2}$$

(Since $d\varphi_q$ is injective this space is an n-dimensional vector subspace of $T_p\mathbb{R}^N$.)

One problem with this definition is that it appears to depend on the choice of φ. To get around this problem, we'll give an alternative definition of T_pX. In §4.1 we showed that there exists a neighborhood, V, of p in $\mathbb{R}^N$ (which we can without loss of generality take to be the same as V above) and a $\mathcal{C}^\infty$ map

$$f : (V, p) \to (\mathbb{R}^k, 0)\,, \quad k = N - n\,, \tag{4.2.3}$$

such that $X \cap V = f^{-1}(0)$ and such that f is a submersion at all points of $X \cap V$, and in particular at p. Thus

$$df_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$$

is surjective, and hence the kernel of df_p has dimension n. Our alternative definition of T_pX is

$$T_pX = \mathrm{k}ernel\, df_p\,. \tag{4.2.4}$$

The spaces (4.2.2) and (4.2.4) are both n-dimensional subspaces of $T_p\mathbb{R}^N$, and we claim that these spaces are the same. (Notice that the definition (4.2.4) of T_pX doesn't depend on φ, so if we can show that these spaces are the same, the definitions (4.2.2) and (4.2.4) will depend *neither* on φ *nor* on f.)

Proof. Since $\varphi(U)$ is contained in $X \cap V$ and $X \cap V$ is contained in $f^{-1}(0)$, $f \circ \varphi = 0$, so by the chain rule

$$df_p \circ d\varphi_q = d(f \circ \varphi)_q = 0\,. \tag{4.2.5}$$

Hence if $\mathrm{v} \in T_p\mathbb{R}^n$ and $w = d\varphi_q(\mathrm{v})$, $df_p(w) = 0$. This shows that the space (4.2.2) is contained in the space (4.2.4). However, these two spaces are n-dimensional so they coincide.

□

From the proof above one can extract a slightly stronger result:

Theorem 4.2.3. *Let W be an open subset of $\mathbb{R}^\ell$ and $h : (W, q) \to (\mathbb{R}^N, p)$ a $\mathcal{C}^\infty$ map. Suppose $h(W)$ is contained in X. Then the image of the map*

$$dh_q : T_q\mathbb{R}^\ell \to T_p\mathbb{R}^N$$

is contained in T_pX.

Proof. Let f be the map (4.2.3). We can assume without loss of generality that $h(W)$ is contained in V, and so, by assumption, $h(W) \subseteq X \cap V$. Therefore, as above, $f \circ h = 0$, and hence $dh_q(T_q\mathbb{R}^\ell)$ is contained in the kernel of df_p.

□

This result will enable us to define the *derivative* of a mapping between manifolds. Explicitly: Let X be a submanifold of $\mathbb{R}^N$, Y a submanifold of $\mathbb{R}^m$ and $g : (X, p) \to (Y, y_0)$ a $\mathcal{C}^\infty$ map. By Definition 4.1.1 there exists a neighborhood, $\mathcal{O}$, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $\widetilde{g} : \mathcal{O} \to \mathbb{R}^m$ extending to g. We will define

$$(dg_p) : T_pX \to T_{y_0}Y \tag{4.2.6}$$

to be the restriction of the map

$$(d\widetilde{g})_p : T_p\mathbb{R}^N \to T_{y_0}\mathbb{R}^m \tag{4.2.7}$$

to T_pX. There are two obvious problems with this definition:

1. Is the space
$$(d\widetilde{g}_p)(T_pX)$$
contained in $T_{y_0}Y$?

2. Does the definition depend on $\widetilde{g}$?

To show that the answer to 1. is yes and the answer to 2. is no, let
$$\varphi : (U, x_0) \to (X \cap V, p)$$
be a parametrization of X, and let $h = \widetilde{g} \circ \varphi$. Since $\varphi(U) \subseteq X$, $h(U) \subseteq Y$ and hence by Theorem 4.2.4
$$dh_{x_0}(T_{x_0}\mathbb{R}^n) \subseteq T_{y_0}Y \,.$$
But by the chain rule

$$dh_{x_0} = d\widetilde{g}_p \circ d\varphi_{x_0} \,, \tag{4.2.8}$$

so by (4.2.2)

$$(d\widetilde{g}_p)(T_pX) \quad \subseteq \quad T_pY \tag{4.2.9}$$

and

$$(d\widetilde{g}_p)(T_pX) \quad = \quad (dh)_{x_0}(T_{x_0}\mathbb{R}^n) \tag{4.2.10}$$

Thus the answer to 1. is yes, and since $h = \widetilde{g} \circ \varphi = g \circ \varphi$, the answer to 2. is no.

From (4.2.5) and (4.2.6) one easily deduces

Theorem 4.2.4 (Chain rule for mappings between manifolds)**.** *Let Z be a submanifold of $\mathbb{R}^\ell$ and $\psi : (Y, y_0) \to (Z, z_0)$ a $\mathcal{C}^\infty$ map. Then $d\psi_{y_0} \circ dg_p = d(\psi \circ g)_p$.*

We will next prove manifold versions of the inverse function theorem and the canonical immersion and submersion theorems.

Theorem 4.2.5 (Inverse function theorem for manifolds)**.** *Let X and Y be n-dimensional manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Suppose that at $p \in X$ the map*
$$df_p : T_pX \to T_qY \,, \quad q = f(p) \,,$$
is bijective. Then f maps a neighborhood, U, of p in X diffeomorphically onto a neighborhood, V, of q in Y.

Proof. Let U and V be open neighborhoods of p in X and q in Y and let

$$\varphi_0 : (U_0, p_0) \to (U, p)$$

and

$$\psi_0 : (V_0, q_0) \to (V, q)$$

be parametrizations of these neighborhoods. Shrinking U_0 and U we can assume that $f(U) \subseteq V$. Let

$$g : (U_0, p_0) \to (V_0, q_0)$$

be the map $\psi_0^{-1} \circ f \circ \varphi_0$. Then $\psi_0 \circ g = f \circ \varphi$, so by the chain rule

$$(d\psi_0)_{q_0} \circ (dg)_{p_0} = (df)_p \circ (d\varphi_0)_{p_0} .$$

Since $(d\psi_0)_{q_0}$ and $(d\varphi_0)_{p_0}$ are bijective it's clear from this identity that if df_p is bijective the same is true for $(dg)_{p_0}$. Hence by the inverse function theorem for open subsets of $\mathbb{R}^n$, g maps a neighborhood of p_0 in U_0 diffeomorphically onto a neighborhood of q_0 in V_0. Shrinking U_0 and V_0 we assume that these neighborhoods are U_0 and V_0 and hence that g is a diffeomorphism. Thus since $f : U \to V$ is the map $\psi_0 \circ g \circ \varphi_0^{-1}$, it is a diffeomorphism as well.

□

Theorem 4.2.6 (The canonical submersion theorem for manifolds)**.** *Let X and Y be manifolds of dimension n and m, $m < n$, and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map. Suppose that at $p \in X$ the map*

$$df_p : T_pX \to T_qY , \quad q = f(p) ,$$

is surjective. Then there exists an open neighborhood, U, of p in X, and open neighborhood, V of $f(U)$ in Y and parametrizations

$$\varphi_0 : (U_0, 0) \to (U, p)$$

and

$$\psi_0 : (V_0, 0) \to (V, q)$$

such that in the diagram below

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle \varphi_0}\big\uparrow & & \big\uparrow{\scriptstyle \psi_0} \\ U_0 & \longrightarrow & V_0 \end{array}$$

the bottom arrow, $\psi_0^{-1} \circ f \circ \varphi_0$*, is the canonical submersion,* π*.*

Proof. Let U and V be open neighborhoods of p and q and

$$\varphi_0 : (U_0, p_0) \to (U, p)$$

and

$$\psi_0 : (V_0, q_0) \to (V, q)$$

be parametrizations of these neighborhoods. Composing φ_0 and ψ_0 with the translations we can assume that p_0 is the origin in $\mathbb{R}^n$ and q_0 the origin in $\mathbb{R}^m$, and shrinking U we can assume $f(U) \subseteq V$. As above let $g : (U_0, 0) \to (V_0, 0)$ be the map, $\psi_0^{-1} \circ f \circ \varphi_0$. By the chain rule

$$(d\psi_0)_0 \circ (dg)_0 = df_p \circ (d\varphi_0)_0 \,,$$

therefore, since $(d\psi_0)_0$ and $(d\varphi_0)_0$ are bijective it follows that $(dg)_0$ is surjective. Hence, by Theorem **??**, we can find an open neighborhood, U, of the origin in $\mathbb{R}^n$ and a diffeomorphism, $\varphi_1 : (U_1, 0) \to (U_0, 0)$ such that $g \circ \varphi_1$ is the canonical submersion. Now replace U_0 by U_1 and φ_0 by $\varphi_0 \circ \varphi_1$.

□

Theorem 4.2.7 (The canonical immersion theorem for manifolds)**.** *Let* X *and* Y *be manifolds of dimension* n *and* m*,* $n < m$*, and* $f : X \to Y$ *a* $\mathcal{C}^\infty$ *map. Suppose that at* $p \in X$ *the map*

$$df_p : T_pX \to T_qY \,, \quad q = f(p)$$

is injective. Then there exists an open neighborhood, U*, of* p *in* X*, an open neighborhood,* V*, of* $f(U)$ *in* Y *and parametrizations*

$$\varphi_0 : (U_0, 0) \to (U, p)$$

and

$$\psi_0 : (V_0, 0) \to (V, q)$$

such that in the diagram below

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & V \\ {\scriptstyle\varphi_0}\Big\uparrow & & \Big\uparrow{\scriptstyle\psi_0} \\ U_0 & \longrightarrow & V_0 \end{array}$$

the bottom arrow, $\psi_0 \circ f \circ \varphi_0$*, is the canonical immersion,* ι*.*

Proof. The proof is identical with the proof of Theorem 4.2.6 except for the last step. In the last step one converts g into the canonical immersion via a map $\psi_1 : (V_1, 0) \to (V_0, 0)$ with the property $g \circ \psi_1 = \iota$ and then replaces ψ_0 by $\psi_0 \circ \psi_1$.

□

Exercises.

1. What is the tangent space to the quadric, $x_n = x_1^2 + \cdots + x_{n-1}^2$, at the point, $(1, 0, \ldots, 0, 1)$?

2. Show that the tangent space to the $(n-1)$-sphere, S^{n-1}, at p, is the space of vectors, $(p, \mathrm{v}) \in T_p\mathbb{R}^n$ satisfying $p \cdot \mathrm{v} = 0$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let $X = \mathrm{graph} f$. What is the tangent space to X at $(a, f(a))$?

4. Let $\sigma : S^{n-1} \to S^{n-1}$ be the antipodal map, $\sigma(x) = -x$. What is the derivative of σ at $p \in S^{n-1}$?

5. Let $X_i \subseteq \mathbb{R}^{N_i}$, $i = 1, 2$, be an n_i-dimensional manifold and let $p_i \in X_i$. Define X to be the Cartesian product

$$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

and let $p = (p_1, p_2)$. Show that T_pX is the vector space sum of the vector spaces,$T_{p_1}X_1$ and $T_{p_2}X_2$.

6. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold and $\varphi_i : U_i \to X \cap V_i$, $i = 1, 2$, two parametrizations. From these parametrizations one gets an overlap diagram

$$\begin{array}{ccccc} & & X \cap V & & \\ & d_1 \nearrow & & \nwarrow d_2 & \\ W_1 & & \xrightarrow{\psi} & & W_2 \end{array} \tag{4.2.11}$$

where $V = V_1 \cap V_2$, $W_i = \varphi_i^{-1}(X \cap V)$ and $\psi = \varphi_2^{-1} \circ \varphi_1$.

(a) Let $p \in X \cap V$ and let $q_i = \varphi_i^{-1}(p)$. Derive from the overlap diagram (4.2.11) an overlap diagram of linear maps

$$\begin{array}{ccccc} & & T_p\mathbb{R}^N & & \\ & (d\varphi_1)_{q_1} \nearrow & & \searrow (d\varphi_2)_{q_2} & \\ T_{q_1}\mathbb{R}^n & & \xrightarrow{(d\psi)_{q_1}} & & T_{q_2}\mathbb{R}^n \end{array} \tag{4.2.12}$$

(b) Use overlap diagrams to give another proof that T_pX is intrinsically defined.

4.3 Vector fields and differential forms on manifolds

A vector field on an open subset, U, of $\mathbb{R}^n$ is a function, v, which assigns to each $p \in U$ an element, $v(p)$, of T_pU, and a k-form is a function, ω, which assigns to each $p \in U$ an element, $\omega(p)$, of $\Lambda^k(T_p^*)$. These definitions have obvious generalizations to manifolds:

Definition 4.3.1. *Let X be a manifold. A vector field on X is a function, v, which assigns to each $p \in X$ an element, $v(p)$, of T_pX, and a k-form is a function, ω, which assigns to each $p \in X$ an element, $\omega(p)$, of $\Lambda^k(T_p^*X)$.*

We'll begin our study of vector fields and k-forms on manifolds by showing that, like their counterparts on open subsets of $\mathbb{R}^n$, they have nice pull-back and push-forward properties with respect to mappings. Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ mapping.

Definition 4.3.2. *Given a vector field, v, on X and a vector field, w, on Y, we'll say that v and w are f-related if, for all $p \in X$ and $q = f(p)$*

$$(df)_p v(p) = w(q)\,. \tag{4.3.1}$$

In particular, if f is a diffeomorphism, and we're given a vector field, v, on X we can define a vector field, w, on Y by requiring that for every point, $q \in Y$, the identity (**??**) holds at the point, $p = f^{-1}(q)$. In this case we'll call w the *push-forward* of v by f and denote it by f_*v. Similarly, given a vector field, w, on Y we can define a vector field, v, on X by applying the same construction to the inverse diffeomorphism, $f^{-1} : Y \to X$. We will call the vector field $(f^{-1})_*w$ the *pull-back* of w by f (and also denote it by f^*w).

For differential forms the situation is even nicer. Just as in §2.5 we can define the pull-back operation on forms for *any* $\mathcal{C}^\infty$ map $f : X \to Y$. Specifically: Let ω be a k-form on Y. For every $p \in X$, and $q = f(p)$ the linear map

$$df_p : T_pX \to T_qY$$

induces by (1.8.2) a pull-back map

$$(df_p)^* : \Lambda^k(T_q^*) \to \Lambda^k(T_p^*)$$

and, as in §2.5, we'll define the pull-back, $f^*\omega$, of ω to X by defining it at p by the identity

$$(f^*\omega)(p) = (df_p)^*\omega(q)\,. \tag{4.3.2}$$

The following results about these operations are proved in exactly the same way as in §2.5.

Proposition 4.3.3. *Let X, Y and Z be manifolds and $f : X \to Y$ and $g : Y \to Z$ $\mathcal{C}^\infty$ maps. Then if ω is a k-form on Z*

$$f^*(g^*\omega) = (g \circ f)^*\omega\,, \tag{4.3.3}$$

and if v is a vector field on X and f and g are diffeomorphisms

$$(g \circ f)_*v = g_*(f_*v)\,. \tag{4.3.4}$$

Our first application of these identities will be to define what one means by a "$\mathcal{C}^\infty$ vector field" and a "$\mathcal{C}^\infty$ k-form".

Let X be an n-dimensional manifold and U an open subset of X.

Definition 4.3.4. *The set U is a* parametrizable *open set if there exists an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\varphi_0 : U_0 \to U$.*

In other words, U is parametrizable if there exists a parametrization having U as its image. (Note that X being a manifold means that every point is contained in a parametrizable open set.)

Now let $U \subseteq X$ be a parametrizable open set and $\varphi : U_0 \to U$ a parametrization of U.

Definition 4.3.5. *A k-form ω on U is $\mathcal{C}^\infty$ if $\varphi_0^*\omega$ is $\mathcal{C}^\infty$.*

This definition appears to depend on the choice of the parametrization, φ, but we claim it doesn't. To see this let $\varphi_1 : U_1 \to U$ be another parametrization of U and let

$$\psi : U_0 \to U_1$$

be the composite map, $\varphi_0^{-1} \circ \varphi_0$. Then $\varphi_0 = \varphi_1 \circ \psi$ and hence by Proposition 4.3.3

$$\varphi_0^*\omega = \psi^*\varphi_1^*\omega \,,$$

so by (2.5.11) $\varphi_0^*\omega$ is $\mathcal{C}^\infty$ if $\varphi_1^*\omega$ is $\mathcal{C}^\infty$. The same argument applied to ψ^{-1} shows that $\varphi_1^*\omega$ is $\mathcal{C}^\infty$ if $\varphi_0^*\omega$ is $\mathcal{C}^\infty$. Q.E.D

The notion of "$\mathcal{C}^\infty$" for vector fields is defined similarly:

Definition 4.3.6. *A vector field, v, on U is $\mathcal{C}^\infty$ if φ_0^*v is $\mathcal{C}^\infty$.*

By Proposition 4.3.3 $\varphi_0^*v = \psi^*\varphi_1^*v$, so, as above, this definition is independent of the choice of parametrization.

We now globalize these definitions.

Definition 4.3.7. *A k-form, ω, on X is $\mathcal{C}^\infty$ if, for every point $p \in X$, ω is $\mathcal{C}^\infty$ on a neighborhood of p. Similarly, a vector field, v, on X is $\mathcal{C}^\infty$ if, for every point, $p \in X$, v is $\mathcal{C}^\infty$ on a neighborhood of p.*

We will also use the identities (4.3.4) and (4.3.5) to prove the following two results.

Proposition 4.3.8. *Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Then if ω is a $\mathcal{C}^\infty$ k-form on Y, $f^*\omega$ is a $\mathcal{C}^\infty$ k-form on X.*

Proof. For $p \in X$ and $q = f(p)$ let $\varphi_0 : U_0 \to U$ and $\psi_0 : V_0 \to V$ be parametrizations with $p \in U$ and $q \in V$. Shrinking U if necessary we can assume that $f(U) \subseteq V$. Let $g : U_0 \to V_0$ be the map, $g = \psi_0^{-1} \circ f \circ \varphi_0$. Then $\psi_0 \circ g = f \circ \varphi_0$, so $g^*\psi_0^*\omega = \varphi_0^*f^*\omega$. Since ω is $\mathcal{C}^\infty$, $\psi_0^*\omega$ is $\mathcal{C}^\infty$, so by (2.5.11) $g^*\psi_0^*\omega$ is $\mathcal{C}^\infty$, and hence, $\varphi_0^*f^*\omega$ is $\mathcal{C}^\infty$. Thus by definition $f^*\omega$ is $\mathcal{C}^\infty$ on U.

□

By exactly the same argument one proves:

Proposition 4.3.9. *If w is a $\mathcal{C}^\infty$ vector field on Y and f is a diffeomorphism, f^*w is a $\mathcal{C}^\infty$ vector field on X.*

Some notation:

1. We'll denote the space of $\mathcal{C}^\infty$ k-forms on X by $\Omega^k(X)$.

2. For $\omega \in \Omega^k(X)$ we'll define the support of ω to be the closure of the set

$$\{p \in X\,,\, \omega(p) \neq 0\}$$

and we'll denote by $\Omega^k_c(X)$ the space of completely supported k-forms.

3. For a vector field, v, on X we'll define the support of v to be the closure of the set

$$\{p \in X\,,\, v(p) \neq 0\}\,.$$

We will now review some of the results about vector fields and the differential forms that we proved in Chapter 2 and show that they have analogues for manifolds.

1. Integral curves

Let $I \subseteq \mathbb{R}$ be an open interval and $\gamma : I \to X$ a $\mathcal{C}^\infty$ curve. For $t_0 \in I$ we will call $\vec{u} = (t_0, 1) \in T_{t_0}\mathbb{R}$ the *unit* vector in $T_{t_0}\mathbb{R}$ and if $p = \gamma(t_0)$ we will call the vector

$$d\gamma_{t_0}(\vec{u}) \in T_pX$$

the *tangent vector* to γ at p. If v is a vector field on X we will say that γ is an *integral curve* of v if for all $t_0 \in I$

$$v(\gamma(t_0)) = d\gamma_{t_0}(\vec{u})\,.$$

Proposition 4.3.10. *Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. If v and w are vector fields on X and Y which are f-related, then integral curves of v get mapped by f onto integral curves of w.*

Proof. If the curve, $\gamma : I \to X$ is an integral curve of v we have to show that $f \circ \gamma : I \to Y$ is an integral curve of w. If $\gamma(t) = p$ and $q = f(p)$ then by the chain rule

$$\begin{aligned} w(q) &= df_p(v(p)) = df_p(d\gamma_t(\vec{u})) \\ &= d(f \circ \gamma)_t(\vec{u})\,. \end{aligned}$$

□

From this result it follows that the local existence, uniqueness and "smooth dependence on initial data" results about vector fields that we described in §2.1 are true for vector fields on manifolds. More explicitly, let U be a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization. Since U_0 is an open subset of $\mathbb{R}^n$ these results are true for the vector field, $w = \varphi_0^* v$ and hence since w and v are φ_0-related they are true for v. In particular

Proposition 4.3.11 (local existence)**.** *For every $p \in U$ there exists an integral curve, $\gamma(t)\,,\ -\epsilon < t < \epsilon$, of v with $\gamma(0) = p$.*

Proposition 4.3.12 (local uniqueness)**.** *Let $\gamma_i : I_i \to U$ $i = 1, 2$ be integral curves of v and let $I = I_1 \cap I_2$. Suppose $\gamma_2(t) = \gamma_1(t)$ for some $t \in I$. Then there exists a unique integral curve, $\gamma : I \cup I_2 \to U$ with $\gamma = \gamma_1$ on I_1 and $\gamma = \gamma_2$ on I_2.*

Proposition 4.3.13 (smooth dependence on initial data)**.** *For every $p \in U$ there exists a neighborhood, $\mathcal{O}$ of p in U, an interval $(-\epsilon, \epsilon)$ and a $\mathcal{C}^\infty$ map, $h : \mathcal{O} \times (-\epsilon, \epsilon) \to U$ such that for every $p \in \mathcal{O}$ the curve*

$$\gamma_p(t) = h(p, t)\,, \quad -\epsilon < t < \epsilon\,,$$

is an integral curve of v with $\gamma_p(0) = p$.

As in Chapter 2 we will say that v is *complete* if, for every $p \in X$ there exists an integral curve, $\gamma(t)$, $-\infty < t < \infty$, with $\gamma(0) = p$. In Chapter 2 we showed that one simple criterium for a vector field to be complete is that it be compactly supported. We will prove that the same is true for manifolds.

Theorem 4.3.14. *If X is compact or, more generally, if v is compactly supported, v is complete.*

Proof. It's not hard to prove this by the same argument that we used to prove this theorem for vector fields on $\mathbb{R}^n$, but we'll give a simpler proof that derives this directly from the $\mathbb{R}^n$ result. Suppose X is a submanifold of $\mathbb{R}^N$. Then for $p \in X$,

$$T_pX \subset T_p\mathbb{R}^N = \{(p, \mathrm{v})\,, \quad \mathrm{v} \in \mathbb{R}^N\}\,,$$

so $v(p)$ can be regarded as a pair, $(p, \mathrm{v}(p))$ where $\mathrm{v}(p)$ is in $\mathbb{R}^N$. Let

$$f_v : X \to \mathbb{R}^N \tag{4.3.5}$$

be the map, $f_v(p) = \mathrm{v}(p)$. It is easy to check that v is $\mathcal{C}^\infty$ if and only if f_v is $\mathcal{C}^\infty$. (See exercise 11.) Hence (see Appendix B) there exists a neighborhood, $\mathcal{O}$ of X and a map $g : \mathcal{O} \to \mathbb{R}^N$ extending f_v. Thus the vector field w on $\mathcal{O}$ defined by $w(q) = (q, g(q))$ extends the vector field v to $\mathcal{O}$. In other words if $\iota : X \hookrightarrow \mathcal{O}$ is the inclusion map, v and w are ι-related. Thus by Proposition 4.3.10 the integral curves of v are just integral curves of w that are contained in X.

Suppose now that v is compactly supported. Then there exists a function $\rho \in \mathcal{C}_o^\infty(\mathcal{O})$ which is 1 on the support of v, so, replacing w by ρw, we can assume that w is compactly supported. Thus w is complete. Let $\gamma(t)$, $-\infty < t < \infty$ be an integral curve of w. We will prove that if $\gamma(0) \in X$, then this curve is an integral curve of v. We first observe:

Lemma 4.3.15. *The set of points, $t \in \mathbb{R}$, for which $\gamma(t) \in X$ is both open and closed.*

Proof. If $p \notin \operatorname{supp} v$ then $w(p) = 0$ so if $\gamma(t) = p$, $\gamma(t)$ is the constant curve, $\gamma = p$, and there's nothing to prove. Thus we are reduced to showing that the set

$$\{t \in \mathbb{R}\,, \quad \gamma(t) \in \operatorname{supp} v\} \tag{4.3.6}$$

is both open and closed. Since $\operatorname{supp} v$ is compact this set is clearly closed. To show that it's open suppose $\gamma(t_0) \in \operatorname{supp} v$. By local existence there exist an interval $(-\epsilon + t_0\,, \epsilon + t_0)$ and an integral curve, $\gamma_1(t)$, of v defined on this interval and taking the value $\gamma_1(t_0) = \gamma(t_0)$ at p. However since v and w are ι-related γ_1 is also an integral curve of w and so it has to coincide with γ on the interval $(-\epsilon + t_0\,, \epsilon + t_0)$. In particular, for t on this interval, $\gamma(t) \in \operatorname{supp} v$, so the set (4.3.6) is open.

□

To conclude the proof of Theorem 4.3.14 we note that since $\mathbb{R}$ is connected it follows that if $\gamma(t_0) \in X$ for some $t_0 \in \mathbb{R}$ then $\gamma(t) \in X$ for *all* $t \in \mathbb{R}$, and hence γ is an integral curve of v. Thus in particular *every* integral curve of v exists for all time, so v is complete.

$\square$

Since w is complete it generates a one-parameter group of diffeomorphisms, $g_t : \mathcal{O} \to \mathcal{O}$, $-\infty < t < \infty$ having the property that the curve

$$g_t(p) = \gamma_p(t)\,, \quad -\infty < t < \infty$$

is the unique integral curve of w with initial point, $\gamma_p(0) = p$. But if $p \in X$ this curve is an integral curve of v, so the restriction

$$f_t = g_t \big| X$$

is a one-parameter group of diffeomorphisms of X with the property that for $p \in X$ the curve

$$f_t(p) = \gamma_p(t)\,, \quad -\infty < t < \infty$$

is the unique integral curve of v with initial point $\gamma_p(0) = p$.

2. The exterior differentiation operation

Let ω be a $\mathcal{C}^\infty$ k-form on X and $U \subset X$ a parametrizable open set. Given a parametrization, $\varphi_0 : U_0 \to U$ we define the exterior derivative, $d\omega$, of ω on X by the formula

$$d\omega = (\varphi_0^{-1})^*\, d\varphi_0^*\omega\,. \tag{4.3.7}$$

(Notice that since U_0 is an open subset of $\mathbb{R}^n$ and $\varphi_0^*\omega$ a k-form on U_0, the "d" on the right is well-defined.) We claim that this definition doesn't depend on the choice of parametrization. To see this let $\varphi_1 : U_1 \to U$ be another parametrization of U and let $\psi : U_0 \to U_1$ be the diffeomorphism, $\varphi_1^{-1} \circ \varphi_0$. Then $\varphi_0 = \varphi_1 \circ \psi$ and hence

$$\begin{aligned} d\varphi_0^*\omega &= d\psi^*\varphi_1^*\omega = \psi^*\, d\varphi_1^*\omega \\ &= \varphi_0^*(\varphi_1^{-1})^*\, d\varphi_1^*\omega \end{aligned}$$

hence

$$(\varphi_0^{-1})^*\, d\varphi_0^*\omega = (\varphi_1^{-1})^*\, d\varphi_1^*\omega$$

as claimed. We can therefore, define the exterior derivative, $d\omega$, globally by defining it to be equal to (4.3.7) on every parametrizable open set.

It's easy to see from the definition (4.3.7) that this exterior differentiation operation inherits from the exterior differentiation operation on open subsets of $\mathbb{R}^n$ the properties (2.3.2) and (2.3.3) and that for zero forms, i.e., $\mathcal{C}^\infty$ functions, $f : X \to \mathbb{R}$, df is the "intrinsic" df defined in Section 2.1, i.e., for $p \in X$ df_p is the derivative of f

$$df_p : T_pX \to \mathbb{R}$$

viewed as an element of $\Lambda^1(T_p^*X)$. Let's check that it also has the property (2.5.12).

Theorem 4.3.16. *Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Then for $\omega \in \Omega^k(Y)$*

$$f^*\, d\omega = df^*\omega\,. \tag{4.3.8}$$

Proof. For every $p \in X$ we'll check that this equality holds in a neighborhood of p. Let $q = f(p)$ and let U and V be parametrizable neighborhoods of p and q. Shrinking U if necessary we can assume $f(U) \subseteq V$. Given parametrizations

$$\varphi : U_0 \to U$$

and

$$\psi : V_0 \to V$$

we get by composition a map

$$g : U_0 \to V_0\,, \quad g = \psi^{-1} \circ f \circ \varphi$$

with the property $\psi \circ g = f \circ \varphi$. Thus

$$\begin{aligned}
\varphi^*\, d(f^*\omega) &= d\varphi^* f^*\omega \quad \text{(by definition of } d\text{)}\\
&= d(f \circ \varphi)^*\omega\\
&= d(\psi \circ g)^*\omega\\
&= dg^*(\psi^*\omega)\\
&= g^*\, d\varphi^*\omega \quad \text{by (2.5.12)}\\
&= g^*\psi^*\, d\omega \quad \text{(by definition of } d\text{)}\\
&= \varphi^* f^*\, d\omega\,.
\end{aligned}$$

Hence $df^*\omega = f^*\, d\omega$.

□

3. The interior product and Lie derivative operation

Given a k-form, $\omega \in \Omega^k(X)$ and a $\mathcal{C}^\infty$vector field, w, we will define the interior product

$$\iota(v)\omega \in \Omega^{k-1}(X)\,, \tag{4.3.9}$$

as in §2.4, by setting

$$(\iota(v)\omega)_p = \iota(v_p)\omega_p$$

and the Lie derivative

$$L_v\omega \quad = \quad \Omega^k(X) \tag{4.3.10}$$

by setting

$$L_v\omega \quad = \quad \iota(v)\,d\omega + d\iota(v)\omega\,. \tag{4.3.11}$$

It's easily checked that these operations satisfy the identities (2.4.2)–(2.4.8) and (2.4.12)–(2.4.13) (since, just as in §2.4, these identities are deduced from the definitions (4.3.9) and (4.3.1) by purely formal manipulations). Moreover, if v is complete and

$$f_t : X \to X\,, \quad -\infty < t < \infty$$

is the one-parameter group of diffeomorphisms of X generated by v the Lie derivative operation can be defined by the alternative recipe

$$L_v\omega = \left(\frac{d}{dt}f_t^*\omega\right)(t=0) \tag{4.3.12}$$

as in (2.5.22). (Just as in §2.5 one proves this by showing that the operation (4.3.12) has the properties (2.12) and (2.13) and hence that it agrees with the operation (4.3.11) provided the two operations agree on zero-forms.)

Exercises.

1. Let $X \subseteq \mathbb{R}^3$ be the paraboloid, $x_3 = x_1^2 + x_2^2$ and let w be the vector field

$$w = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3}\,.$$

(a) Show that w is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

2. Let S^2 be the unit 2-sphere, $x_1^2 + x_2^2 + x_3^2 = 1$, in $\mathbb{R}^3$ and let w be the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\,.$$

(a) Show that w is tangent to S^2, and hence by restriction defines a vector field, v, on S^2.

(b) What are the integral curves of v?

3. As in problem 2 let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and let w be the vector field

$$w = \frac{\partial}{\partial x_3} - x_3 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right)$$

(a) Show that w is tangent to S^2 and hence by restriction defines a vector field, v, on S^2.

(b) What do its integral curves look like?

4. Let S^1 be the unit circle, $x_1^2 + x_2^2 = 1$, in $\mathbb{R}^2$ and let $X = S^1 \times S^1$ in $\mathbb{R}^4$ with defining equations

$$\begin{aligned} f_1 &= x_1^2 + x_2^2 - 1 = 0 \\ f_2 &= x_3^2 + x_4^2 - 1 = 0\,. \end{aligned}$$

(a) Show that the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \lambda \left(x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right)\,,$$

$\lambda \in \mathbb{R}$, is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

(c) Show that $L_w f_i = 0$.

5. For the vector field, v, in problem 4, describe the one-parameter group of diffeomorphisms it generates.

6. Let X and v be as in problem 1 and let $f : \mathbb{R}^2 \to X$ be the map, $f(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$. Show that if u is the vector field,

$$u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \,,$$

then $f_* u = v$.

7. Let X be a submanifold of X in $\mathbb{R}^N$ and let v and w be the vector fields on X and U. Denoting by ι the inclusion map of X into U, show that v and w are ι-related if and only if w is tangent to X and its restriction to X is v.

8. Let X be a submanifold of $\mathbb{R}^N$ and U an open subset of $\mathbb{R}^N$ containing X, and let v and w be the vector fields on X and U. Denoting by ι the inclusion map of X into U, show that v and w are ι-related if and only if w is tangent to X and its restriction to X is v.

9. An elementary result in number theory asserts

Theorem 4.3.17. *A number, $\lambda \in \mathbb{R}$, is irrational if and only if the set*

$$\{m + \lambda n \,, \quad m \text{ and } n \text{ intgers}\}$$

is a dense *subset of* $\mathbb{R}$.

Let v be the vector field in problem 4. Using the theorem above prove that if $\lambda/2\pi$ is irrational then for every integral curve, $\gamma(t)$, $-\infty < t < \infty$, of v the set of points on this curve is a dense subset of X.

10. Let X be an n-dimensional submanifold of $\mathbb{R}^N$. Prove that a vector field, v, on X is $\mathcal{C}^\infty$ if and only if the map, (4.3.5) is $\mathcal{C}^\infty$.

Hint: Let U be a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization of U. Composing φ with the inclusion map $\iota : X \to \mathbb{R}^N$ one gets a map, $\iota \circ \varphi : U \to \mathbb{R}^N$. Show that if

$$\varphi^* v \quad = \quad \sum v_i \frac{\partial}{\partial x_j}$$

then

$$\varphi^* f_i = \sum \frac{\partial \varphi_i}{\partial x_j} v_j$$

where $f_1, \dots, f_N$ are the coordinates of the map, f_v, and $\varphi_1, \dots, \varphi_N$ the coordinates of $\iota \circ \varphi$.

11. Let v be a vector field on X and $\varphi : X \to \mathbb{R}$, a $\mathcal{C}^\infty$ function. Show that if the function

$$L_v \varphi = \iota(v)\, d\varphi \tag{4.3.13}$$

is zero φ is constant along integral curves of v.

12. Suppose that $\varphi : X \to \mathbb{R}$ is proper. Show that if $L_v \varphi = 0$, v is complete.

Hint: For $p \in X$ let $a = \varphi(p)$. By assumption, $\varphi^{-1}(a)$ is compact. Let $\rho \in \mathcal{C}_0^\infty(X)$ be a "bump" function which is one on $\varphi^{-1}(a)$ and let w be the vector field, ρv. By Theorem 4.3.14, w is complete and since

$$L_w \varphi = \iota(\rho v)\, d\varphi = \rho \iota(v)\, d\varphi = 0$$

φ is constant along integral curves of w. Let $\gamma(t)$, $-\infty < t < \infty$, be the integral curve of w with initial point, $\gamma(0) = p$. Show that γ is an integral curve of v.

4.4 Orientations

The last part of Chapter 4 will be devoted to the "integral calculus" of forms on manifolds. In particular we will prove manifold versions of two basic theorems of integral calculus on $\mathbb{R}^n$, Stokes theorem and the divergence theorem, and also develop a manifold version of degree theory. However, to extend the integral calculus to manifolds without getting involved in horrendously technical "orientation" issues we will confine ourselves to a special class of manifolds: orientable manifolds. The goal of this section will be to explain what this term means.

Definition 4.4.1. *Let X be an n-dimensional manifold. An* orientation *of X is a rule for assigning to each $p \in X$ an orientation of T_pX.*

Thus by definition 1.9.1 one can think of an orientation as a "labeling" rule which, for every $p \in X$, labels one of the two components of the set, $\Lambda^n(T_p^*X) - \{0\}$, by $\Lambda^n(T_p^*X)_+$, which we'll henceforth call the "plus" part of $\Lambda^n(T_p^*X)$, and the other component by $\Lambda^n(T_p^*X)_-$, which we'll henceforth call the "minus" part of $\Lambda^n(T_p^*X)$.

Definition 4.4.2. *An orientation of X is* smooth *if, for every $p \in X$, there exists a neighborhood, U, of p and a non-vanishing n-form, $\omega \in \Omega^n(U)$ with the property*

$$\omega_q = \Lambda^n(T_q^*X)_+ \tag{4.4.1}$$

for every $q \in U$.

Remark 4.4.3. *If we're given an orientation of X we can define another orientation by assigning to each $p \in X$ the opposite orientation to the orientation we already assigned, i.e., by switching the labels on $\Lambda^n(T_p^*)_+$ and $\Lambda^n(T_p^*)_-$. We will call this the* reversed *orientation of X. We will leave for you to check as an exercise that if X is connected and equipped with a smooth orientation, the only smooth orientations of X are the given orientation and its reversed orientation.*

Hint: *Given any smooth orientation of X the set of points where it agrees with the given orientation is open, and the set of points where it doesn't is also open. Therefore one of these two sets has to be empty.*

Note that if $\omega \in \Omega^n(X)$ is a non-vanishing n-form one gets from ω a smooth orientation of X by requiring that the "labeling rule" above satisfy

$$\omega_p \in \Lambda^n(T_p^*X)_+ \tag{4.4.2}$$

for every $p \in X$. If ω has this property we will call ω a *volume form.* It's clear from this definition that if ω_1 and ω_2 are volume forms on X then $\omega_2 = f_{2,1}\omega_1$ where $f_{2,1}$ is an everywhere positive $\mathcal{C}^\infty$ function.

Example 1.

Open subsets, U of $\mathbb{R}^n$. We will usually assign to U its *standard orientation*, by which we will mean the orientation defined by the n-form, $dx_1 \wedge \cdots \wedge dx_n$.

Example 2.

Let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map. If zero is a regular value of f, the set $X = f^{-1}(0)$ is a submanifold of $\mathbb{R}^N$ of dimension, $n = N - k$, by Theorem **??**. Moreover, for $p \in X$, T_pX is the kernel of the surjective map

$$df_p : T_p\mathbb{R}^N \to T_o\mathbb{R}^k$$

so we get from df_p a bijective linear map

$$T_p\mathbb{R}^N/T_pX \to T_o\mathbb{R}^k\,. \tag{4.4.3}$$

As explained in example 1, $T_p\mathbb{R}^N$ and $T_o\mathbb{R}^k$ have "standard" orientations, hence if we require that the map (4.4.3) be orientation preserving, this gives $T_p\mathbb{R}^N/T_pX$ an orientation and, by Theorem 1.9.4, gives T_pX an orientation. It's intuitively clear that since df_p varies smoothly with respect to p this orientation does as well; however, this fact requires a proof, and we'll supply a sketch of such a proof in the exercises.

Example 3.

A special case of example 2 is the n-sphere

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\,,\, x_1^2 + \cdots + x_{n+1}^2 = 1\}\,,$$

which acquires an orientation from its defining map, $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1$.

Example 4.

Let X be an oriented submanifold of $\mathbb{R}^N$. For every $p \in X$, T_pX sits inside $T_p\mathbb{R}^N$ as a vector subspace, hence, via the identification, $T_p\mathbb{R}^N \leftrightarrow \mathbb{R}^N$ one can think of T_pX as a vector subspace of $\mathbb{R}^N$. In particular from the standard Euclidean inner product on $\mathbb{R}^N$ one gets, by restricting this inner product to vectors in T_pX, an inner product,

$$B_p : T_pX \times T_pX \to \mathbb{R}$$

on T_pX. Let σ_p be the volume element in $\Lambda^n(T_p^*X)$ associated with B_p (see §1.9, exercise 10) and let $\sigma = \sigma_X$ be the non-vanishing n-form on X defined by the assignment

$$p \in X \to \sigma_p \,.$$

In the exercises at the end of this section we'll sketch a proof of the following.

Theorem 4.4.4. *The form, σ_X, is $\mathcal{C}^\infty$ and hence, in particular, is a volume form. (We will call this form the* Riemannian *volume form.)*

Example 5. The Möbius strip. The Möbius strip is a surface in $\mathbb{R}^3$ which is *not* orientable. It is obtained from the rectangle

$$R = \{(x,y)\,;\, 0 \leq x \leq 1\,,\, -1 < y < 1\}$$

by gluing the ends together in the wrong way, i.e., by gluing $(1,y)$ to $(0,-y)$. It is easy to see that the Möbius strip can't be oriented by taking the standard orientation at $p=(1,0)$ and moving it along the line, $(t,0)$, $0 \leq t \leq 1$ to the point, $(0,0)$ (which is *also* the point, p, after we've glued the ends of the rectangle together).

We'll next investigate the "compatibility" question for diffeomorphisms between oriented manifolds. Let X and Y be n-dimensional manifolds and $f : X \to Y$ a diffeomorphism. Suppose both of these manifolds are equipped with orientations. We will say that f is *orientation preserving* if, for all $p \in X$ and $q = f(p)$ the linear map

$$df_p : T_pX \to T_qY$$

is orientation preserving. It's clear that if ω is a volume form on Y then f is orientation preserving if and only if $f^*\omega$ is a volume form on X, and from (1.9.5) and the chain rule one easily deduces

Theorem 4.4.5. *If Z is an oriented n-dimensional manifold and $g : Y \to Z$ a diffeomorphism, then if both f and g are orientation preserving, so is $g \circ f$.*

If $f : X \to Y$ is a diffeomorphism then the set of points, $p \in X$, at which the linear map,

$$df_p : T_pX \to T_qY \,, \quad q = f(p)\,,$$

is orientation preserving is open, and the set of points at which its orientation reversing is open as well. Hence if X is connected, df_p has to be orientation preserving at all points or orientation reversing at all points. In the latter case we'll say that f is *orientation reversing.*

If U is a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization of U we'll say that this parametrization is an *oriented* parametrization if φ is orientation preserving with respect to the standard orientation of U_0 and the given orientation on U. Notice that if this parametrization isn't oriented we can convert it into one that is by replacing every connected component, V_0, of U_0 on which φ isn't orientation preserving by the open set

$$V_0^\sharp = \{(x_1, \dots, x_n) \in \mathbb{R}^n\,, \quad (x_1, \dots, x_{n1}, -x_n) \in V_0\} \tag{4.4.4}$$

and replacing φ by the map

$$\psi(x_1, \dots, x_n) = \varphi(x_1, \dots, x_{n-1}, -x_n)\,. \tag{4.4.5}$$

If $\varphi_i : U_i \to U$, $i = 0, 1$, are oriented parametrizations of U and $\psi : U_0 \to U_1$ is the diffeomorphism, $\varphi_1^{-1} \circ \varphi_0$, then by the theorem above ψ is orientation preserving or in other words

$$\det \left[\frac{\partial \psi_i}{\partial x_j} \right] > 0 \tag{4.4.6}$$

at every point on U_0.

We'll conclude this section by discussing some orientation issues which will come up when we discuss Stokes theorem and the divergence theorem in §4.6. First a definition.

Definition 4.4.6. *An open subset, D, of X is a* smooth domain *if*

> *(a) its boundary is an $(n-1)$-dimensional submanifold of X and*
>
> *(b) the boundary of D coincides with the boundary of the closure of D.*

Examples.

1. The n-ball, $x_1^2 + \cdots + x_n^2 < 1$, whose boundary is the sphere, $x_1^2 + \cdots + x_n^2 = 1$.

2. The n-dimensional annulus,

$$1 < x_1^2 + \cdots + x_n^2 < 2$$

whose boundary consists of the spheres,

$$x_1^2 + \cdots + x_n^2 = 1 \text{ and } x_1^2 + \cdots + x_n^2 = 2\,.$$

3. Let S^{n-1} be the unit sphere, $x_1^2 + \cdots + x_2^2 = 1$ and let $D = \mathbb{R}^n - S^{n-1}$. Then the boundary of D is S^{n-1} but D is not a smooth domain since the boundary of its closure is empty.

4. The simplest example of a smooth domain is the half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\,, \quad x_1 < 0\} \tag{4.4.7}$$

whose boundary

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n\,, \quad x_1 = 0\} \tag{4.4.8}$$

we can identify with $\mathbb{R}^{n-1}$ via the map,

$$(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \to (0, x_2, \ldots, x_n)\,.$$

We will show that every bounded domain looks locally like this example.

Theorem 4.4.7. *Let D be a smooth domain and p a boundary point of D. Then there exists a neighborhood, U, of p in X, an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\psi : U_0 \to U$ such that ψ maps $U_0 \cap \mathbb{H}^n$ onto $U \cap D$.*

Proof. Let Z be the boundary of D. First we will prove:

Lemma 4.4.8. *For every $p \in Z$ there exists an open set, U, in X containing p and a parametrization*

$$\psi : U_0 \to U \tag{4.4.9}$$

of U with the property

$$\psi(U_0 \cap Bd\mathbb{H}^n) = U \cap Z\,. \tag{4.4.10}$$

Proof. X is locally diffeomorphic at p to an open subset of $\mathbb{R}^n$ so it suffices to prove this assertion for X equal to $\mathbb{R}^n$. However, if Z is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ then by **??** there exists, for every $p \in Z$ a neighborhood, U, of p in $\mathbb{R}^n$ and a function, $\varphi \in \mathcal{C}^\infty(U)$ with the properties

$$x \in U \cap Z \Leftrightarrow \varphi(x) = 0 \tag{4.4.11}$$

and

$$d\varphi_p \neq 0\,. \tag{4.4.12}$$

Without loss of generality we can assume by (4.4.12) that

$$\frac{\partial \varphi}{\partial x_1}(p) \neq 0\,. \tag{4.4.13}$$

Hence if $\rho : U \to \mathbb{R}^n$ is the map

$$\rho(x_1, \ldots, x_n) = (\varphi(x), x_2, \ldots, x_n) \tag{4.4.14}$$

$(d\rho)_p$ is bijective, and hence ρ is locally a diffeomorphism at p. Shrinking U we can assume that ρ is a diffeomorphism of U onto an open set, U_0. By (4.4.11) and (4.4.14) ρ maps $U \cap Z$ onto $U_0 \cap Bd\mathbb{H}^n$ hence if we take ψ to be ρ^{-1}, it will have the property (4.4.10).

□

We will now prove Theorem 4.4.4. Without loss of generality we can assume that the open set, U_0, in Lemma 4.4.8 is an open ball with center at $q \in Bd\mathbb{H}^n$ and that the diffeomorphism, ψ maps q to p. Thus for $\psi^{-1}(U \cap D)$ there are three possibilities.

i. $\psi^{-1}(U \cap D) = (\mathbb{R}^n - Bd\mathbb{H}^n) \cap U_0$.

ii. $\psi^{-1}(U \cap D) = (\mathbb{R}^n - \overline{\mathbb{H}}^n) \cap U_0$.

or

iii. $\psi^{-1}(U \cap D) = \mathbb{H}^n \cap U_0$.

However, i. is excluded by the second hypothesis in Definition 4.4.6 and if ii. occurs we can rectify the situation by composing φ with the map, $(x_1, \ldots, x_n) \to (-x_1, x_2, \ldots, x_n)$.

□

Definition 4.4.9. *We will call an open set, U, with the properties above a D-adapted parametrizable open set.*

We will now show that if X is oriented and $D \subseteq X$ is a smooth domain then the boundary, Z, of D acquires from X a natural orientation. To see this we first observe

Lemma 4.4.10. *The diffeomorphism, $\psi : U_0 \to U$ in Theorem 4.4.7 can be chosen to be orientation preserving.*

Proof. If it is not, then by replacing ψ with the diffeomorphism, $\psi^\sharp(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_{n-1}, -x_n)$, we get a D-adapted parametrization of U which *is* orientation preserving. (See (4.4.4)–(4.4.5).) □

Let $V_0 = U_0 \cap \mathbb{R}^{n-1}$ be the boundary of $U_0 \cap \mathbb{H}^n$. The restriction of ψ to V_0 is a diffeomorphism of V_0 onto $U \cap Z$, and we will orient $U \cap Z$ by requiring that this map be an oriented parametrization. To show that this is an "intrinsic" definition, i.e., doesn't depend on the choice of ψ, we'll prove

Theorem 4.4.11. *If $\psi_i : U_i \to U$, $i = 0, 1$, are oriented parametrizations of U with the property*

$$\psi_i : U_i \cap \mathbb{H}^n \to U \cap D$$

the restrictions of ψ_i to $U_i \cap \mathbb{R}^{n-1}$ induce compatible orientations on $U \cap X$.

Proof. To prove this we have to prove that the map, $\varphi_1^{-1} \circ \varphi_0$, restricted to $U \cap Bd\mathbb{H}^n$ is an orientation preserving diffeomorphism of $U_0 \cap \mathbb{R}^{n-1}$ onto $U_1 \cap \mathbb{R}^{n-1}$. Thus we have to prove the following:

Proposition 4.4.12. *Let U_0 and U_1 be open subsets of $\mathbb{R}^n$ and $f : U_0 \to U_1$ an orientation preserving diffeomorphism which maps $U_0 \cap \mathbb{H}^n$ onto $U_1 \cap \mathbb{H}^n$. Then the restriction, g, of f to the boundary, $U_0 \cap \mathbb{R}^{n-1}$, of $U_0 \cap \mathbb{H}^n$ is an orientation preserving diffeomorphism, $g : U_0 \cap \mathbb{R}^{n-1} \to U_1 \cap \mathbb{R}^{n-1}$.*

Let $f(x) = (f_1(x), \ldots, f_n(x))$. By assumption $f_1(x_1, \ldots, x_n)$ is less than zero if x_1 is less than zero and equal to zero if x_1 is equal to zero, hence

$$\frac{\partial f_1}{\partial x_1}(0, x_2, \ldots, x_n) \geq 0 \tag{4.4.15}$$

and

$$\frac{\partial f_1}{\partial x_i}(0, x_2, \ldots, x_n) = 0\,, \quad i > 1 \tag{4.4.16}$$

Moreover, since g is the restriction of f to the set $x_1 = 0$

$$\frac{\partial f_i}{\partial x_j}(0, x_2, \ldots, x_n) = \frac{\partial g_i}{\partial x_j}(x_2, \ldots, x_1) \tag{4.4.17}$$

for $i, j \geq 2$. Thus on the set, $x_1 = 0$

$$\det \left[\frac{\partial f_i}{\partial x_j}\right] = \frac{\partial f_1}{\partial x_1} \det \left[\frac{\partial g_i}{\partial x_j}\right] . \tag{4.4.18}$$

Since f is orientation preserving the left hand side of (4.4.18) is positive at all points $(0, x_2, \ldots, x_n) \in U_0 \cap \mathbb{R}^{n-1}$ hence by (4.4.15) the same is true for $\frac{\partial f_1}{\partial x_1}$ and $\det\left[\frac{\partial g_i}{\partial x_j}\right]$. Thus g is orientation preserving.

Remark 4.4.13. *For an alternative proof of this result see exercise 8 in §3.2 and exercises 4 and 5 in §3.6.*

We will now orient the boundary of D by requiring that for every D-adapted parametrizable open set, U, the orientation of Z coincides with the orientation of $U \cap Z$ that we described above. We will conclude this discussion of orientations by proving a global version of Proposition 4.4.12.

Proposition 4.4.14. *Let X_i, $i = 1, 2$, be an oriented manifold, $D_i \subseteq X_i$ a smooth domain and Z_i its boundary. Then if f is an orientation preserving diffeomorphism of (X_1, D_1) onto (X_2, D_2) the restriction, g, of f to Z_1 is an orientation preserving diffeomorphism of Z_1 onto Z_2.*

Let U be an open subset of X_1 and $\varphi : U_0 \to U$ an oriented D_1-compatible parametrization of U. Then if $V = f(U)$ the map $f \circ \varphi : U \to V$ is an oriented D_2-compatible parametrization of V and hence $g : U \cap Z_1 \to V \cap Z_2$ is orientation preserving.

□

Exercises.

1. Let V be an oriented n-dimensional vector space, B an inner product on V and $e_i \in V$, $i = 1, \dots, n$ an oriented orthonormal basis. Given vectors, $v_i \in V$, $i = 1, \dots, n$ show that if

$$b_{i,j} \quad = \quad B(v_i, v_j) \tag{4.4.19}$$

and

$$v_i \quad = \quad \sum a_{j,i} e_j \,, \tag{4.4.20}$$

the matrices $\mathcal{A} = [a_{i,j}]$ and $\mathcal{B} = [b_{i,j}]$ satisfy the identity:

$$\mathcal{B} = \mathcal{A}^t \mathcal{A} \tag{4.4.21}$$

and conclude that $\det \mathcal{B} = (\det \mathcal{A})^2$. (In particular conclude that $\det \mathcal{B} > 0$.)

2. Let V and W be oriented n-dimensional vector spaces. Suppose that each of these spaces is equipped with an inner product, and let $e_i \in V$, $i = 1, \dots, n$ and $f_i \in W$, $i = 1, \dots, n$ be oriented orthonormal bases. Show that if $A : W \to V$ is an orientation preserving linear mapping and $Af_i = v_i$ then

$$A^* \operatorname{vol}_V = (\det[b_{i,j}])^{\frac{1}{2}} \operatorname{vol}_W \tag{4.4.22}$$

where $\operatorname{vol}_V = e_1^* \wedge \cdots \wedge e_n^*$, $\operatorname{vol}_W = f_1^* \wedge \cdots \wedge f_n^*$ and $[b_{i,j}]$ is the matrix (4.4.19).

3. Let X be an oriented n-dimensional submanifold of $\mathbb{R}^n$, U an open subset of X, U_0 an open subset of $\mathbb{R}^n$ and $\varphi : U_0 \to U$ an oriented parametrization. Let φ_i, $i = 1, \dots, N$, be the coordinates of the map

$$U_0 \to U \hookrightarrow \mathbb{R}^N \,.$$

the second map being the inclusion map. Show that if σ is the Riemannian volume form on X then

$$\varphi^* \sigma \quad = \quad (\det[\varphi_{i,j}])^{\frac{1}{2}} \; dx_1 \wedge \cdots \wedge dx_n \tag{4.4.23}$$

where

$$\varphi_{i,j} \quad = \quad \sum_{k=1}^{N} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j} \quad 1 \leq i, j \leq n \,. \tag{4.4.24}$$

(*Hint:* For $p \in U_0$ and $q = \varphi(p)$ apply exercise 2 with $V = T_qX$, $W = T_p\mathbb{R}^n$, $A = (d\varphi)_p$ and $v_i = (d\varphi)_p \left(\frac{\partial}{\partial x_i}\right)_p$.) Conclude that σ is a $\mathcal{C}^\infty$ infinity n-form and hence that it *is* a volume form.

4. Given a $\mathcal{C}^\infty$ function $f : \mathbb{R} \to \mathbb{R}$, its graph

$$X = \{(x, f(x)), \quad x \in \mathbb{R}\}$$

is a submanifold of $\mathbb{R}^2$ and

$$\varphi : \mathbb{R} \to X, \quad x \to (x, f(x))$$

is a diffeomorphism. Orient X by requiring that φ be orientation preserving and show that if σ is the Riemannian volume form on X then

$$\varphi^*\sigma = \left(1 + \left(\frac{df}{dx}\right)^2\right)^{\frac{1}{2}} dx\,. \tag{4.4.25}$$

Hint: Exercise 3.

5. Given a $\mathcal{C}^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ its graph

$$X = \{(x, f(x)), \quad x \in \mathbb{R}^n\}$$

is a submanifold of $\mathbb{R}^{n+1}$ and

$$\varphi : \mathbb{R}^n \to X, \quad x \to (x, f(x)) \tag{4.4.26}$$

is a diffeomorphism. Orient X by requiring that φ is orientation preserving and show that if σ is the Riemannian volume form on X then

$$\varphi^*\sigma = \left(1 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{\frac{1}{2}} dx_1 \wedge \cdots \wedge dx_n\,. \tag{4.4.27}$$

Hints:

(a) Let $v = (c_1, \ldots, c_n) \in \mathbb{R}^n$. Show that if $C : \mathbb{R}^n \to \mathbb{R}$ is the linear mapping defined by the matrix $[c_i c_j]$ then $Cv = (\sum c_i^2)v$ and $Cw = 0$ if $w \cdot v = 0$.

(b) Conclude that the eigenvalues of C are $\lambda_1 = \sum c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 0$.

(c) Show that the determinant of $I + C$ is $1 + \sum c_i^2$.

(d) Use (a)–(c) to compute the determinant of the matrix (4.4.24) where φ is the mapping (4.4.26).

6. Let V be an oriented N-dimensional vector space and $\ell_i \in V^*$, $i = 1, \ldots, k$, k linearly independent vectors in V^*. Define

$$L : V \to \mathbb{R}^k$$

to be the map $v \to (\ell_1(v), \ldots, \ell_k(v))$.

(a) Show that L is surjective and that the kernel, W, of L is of dimension $n = N - k$.

(b) Show that one gets from this mapping a bijective linear mapping

$$V/W \to \mathbb{R}^K \tag{4.4.28}$$

and hence from the standard orientation on $\mathbb{R}^k$ an induced orientation on V/W and on W. *Hint:* §1.2, exercise 8 and Theorem 1.9.4.

(c) Let ω be an element of $\Lambda^N(V^*)$. Show that there exists a $\mu \in \Lambda^n(V^*)$ with the property

$$\ell_1 \wedge \cdots \wedge \ell_k \wedge \mu = \omega\,. \tag{4.4.29}$$

Hint: Choose an oriented basis, $e_1, \ldots, e_N$ of V such that $\omega = e_1^* \wedge \cdots \wedge e_N^*$ and $\ell_i = e_i^*$ for $i = 1, \ldots, k$, and let $\mu = e_{i+1}^* \wedge \cdots \wedge e_N^*$.

(d) Show that if ν is an element of $\Lambda^n(V^*)$ with the property

$$\ell_1 \wedge \cdots \wedge \ell_k \wedge \nu = 0$$

then there exist elements, ν_i, of $\Lambda^{n-1}(V^*)$ such that

$$\nu = \sum \ell_i \wedge \nu_i\,. \tag{4.4.30}$$

Hint: Same hint as in part (c).

(e) Show that if $\mu = \mu_i$, $i = 1, 2$, are elements of $\Lambda^n(V^*)$ with the property (4.4.29) and $\iota : W \to V$ is the inclusion map then $\iota^*\mu_1 = \iota^*\mu_2$. *Hint:* Let $\nu = \mu_1 - \mu_2$. Conclude from part (d) that $\iota^*\nu = 0$.

(f) Conclude that if μ is an element of $\Lambda^n(V^*)$ satisfying (4.4.29) the element, $\sigma = \iota^*\mu$, of $\Lambda^n(W^*)$ is *intrinsically* defined independent of the choice of μ.

(g) Show that σ lies in $\Lambda^n(V^*)_+$.

7. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map. If zero is a regular value of f, the set, $X = f^{-1}(0)$ is a manifold of dimension $n = N - k$. Show that this manifold has a natural smooth orientation. Some suggestions:

(a) Let $f = (f_1, \ldots, f_k)$ and let

$$df_1 \wedge \cdots \wedge df_k = \sum f_I \, dx_I$$

summed over multi-indices which are strictly increasing. Show that for every $p \in X$ $f_I(p) \neq 0$ for some multi-index, $I = (i_1, \ldots, i_k)$, $1 \le i_1 < \cdots < i_k \le N$.

(b) Let $J = (j_1, \ldots, j_n)$, $1 \le j_1 < \cdots < j_n \le N$ be the complementary multi-index to I, i.e., $j_r \neq i_s$ for all r and s. Show that

$$df_1 \wedge \cdots \wedge df_k \wedge dx_J = \pm f_I \, dx_1 \wedge \cdots \wedge dx_N$$

and conclude that the n-form

$$\mu = \pm \frac{1}{f_I} \, dx_J$$

is a $\mathcal{C}^\infty$ n-form on a neighborhood of p in U and has the property:

$$df_1 \wedge \cdots \wedge df_k \wedge \mu = dx_1 \wedge \cdots \wedge dx_N \,. \tag{4.4.31}$$

(c) Let $\iota : X \to U$ be the inclusion map. Show that the assignment

$$p \in X \to (\iota^*\mu)_p$$

defines an *intrinsic* nowhere vanishing n-form

$$\sigma \in \Omega^n(X)$$

on X. *Hint:* Exercise 6.

(d) Show that the orientation of X defined by σ coincides with the orientation that we described earlier in this section. *Hint:* Same hint as above.

8. Let S^n be the n-sphere and $\iota : S^n \to \mathbb{R}^{n+1}$ the inclusion map. Show that if $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form, $\omega = \sum(-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \ldots dx_{n+1}$, the n-form $\iota^*\omega \in \Omega^n(S^n)$ is the Riemannian volume form.

9. Let S^{n+1} be the $(n+1)$-sphere and let

$$S^{n+1}_+ = \{(x_1, \ldots, x_{n+2}) \in S^{n+1}, \quad x_1 < 0\}$$

be the lower hemi-sphere in S^{n+1}.

(a) Prove that S^{n+1}_+ is a smooth domain.

(b) Show that the boundary of S^{n+1}_+ is S^n.

(c) Show that the boundary orientation of S^n agrees with the orientation of S^n in exercise 8.

4.5 Integration of forms over manifolds

In this section we will show how to integrate differential forms over manifolds. In what follows X will be an oriented n-dimensional manifold and W an open subset of X, and our goal will be to make sense of the integral

$$\int_W \omega \tag{4.5.1}$$

where ω is a compactly supported n-form. We'll begin by showing how to define this integral when the support of ω is contained in a parametrizable open set, U. Let U_0 be an open subset of $\mathbb{R}^n$ and $\varphi_0 : U_0 \to U$ a parametrization. As we noted in §4.4 we can assume without loss of generality that this parametrization is oriented. Making this assumption, we'll define

$$\int_W \omega = \int_{W_0} \varphi_0^* \omega \tag{4.5.2}$$

where $W_0 = \varphi_0^{-1}(U \cap W)$. Notice that if $\varphi^*\omega = f dx_1 \wedge \cdots \wedge dx_n$, then, by assumption, f is in $\mathcal{C}_0^\infty(U_0)$. Hence since

$$\int_{W_0} \varphi_0^*\omega = \int_{W_0} f dx_1 \dots dx_n$$

and since f is a bounded continuous function and is compactly supported the Riemann integral on the right is well-defined. (See Appendix B.) Moreover, if $\varphi_1 : U_1 \to U$ is another oriented parametrization of U and $\psi : U_0 \to U_1$ is the map, $\psi = \varphi_1^{-1} \circ \varphi_0$ then $\varphi_0 = \varphi_1 \circ \psi$, so by Proposition 4.3.3

$$\varphi_0^*\omega = \psi^*\varphi_1^*\omega \,.$$

Moreover, by (4.3.5) ψ is orientation preserving. Therefore since

$$W_1 = \psi(W_0) = \varphi_1^{-1}(U \cap W)$$

Theorem 3.5.2 tells us that

$$\int_{W_1} \varphi_1^*\omega = \int_{W_0} \varphi_0^*\omega \,. \tag{4.5.3}$$

Thus the definition (4.5.2) is a legitimate definition. It doesn't depend on the parametrization that we use to define the integral on the right. From the usual additivity properties of the Riemann integral one gets analogous properties for the integral (4.5.2). Namely for $\omega_i \in \Omega_c^n(U)$, $i = 1, 2$

$$\int_W \omega_1 + \omega_2 = \int_W \omega_1 + \int_W \omega_2 \tag{4.5.4}$$

and for $\omega \in \Omega_c^n(U)$ and $c \in \mathbb{R}$

$$\int_W c\omega = c \int_W \omega \,. \tag{4.5.5}$$

We will next show how to define the integral (4.5.1) for *any* compactly supported n-form. This we will do in more or less the same way that we defined improper Riemann integrals in Appendix B: by using partitions of unity. We'll begin by deriving from the partition of unity theorem in Appendix B a manifold version of this theorem.

Theorem 4.5.1. *Let*

$$\mathbb{U} = \{U_\alpha\,,\ \alpha \in \mathcal{I}\} \tag{4.5.6}$$

be a covering of X be open subsets. Then there exists a family of functions, $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, 3, \ldots$, with the properties

> *(a)* $\rho_i \geq 0$.
>
> *(b) For every compact set, $C \subseteq X$ there exists a positive integer N such that if $i > N$,* $\operatorname{supp} \rho_i \cap C = \emptyset$.
>
> *(c)* $\sum \rho_i = 1$.
>
> *(d) For every i there exists an $\alpha \in \mathcal{I}$ such that* $\operatorname{supp} \rho_i \subseteq U_\alpha$.

Remark 4.5.2. *Conditions (a)–(c) say that the ρ_i's are a partition of unity and (d) says that this partition of unity is subordinate to the covering (4.5.6).*

Proof. To simplify the proof a bit we'll assume that X is a closed subset of $\mathbb{R}^N$. For each U_α choose an open subset, $\mathcal{O}_\alpha$ in $\mathbb{R}^N$ with

$$U_\alpha = \mathcal{O}_\alpha \cap X \tag{4.5.7}$$

and let $\mathcal{O}$ be the union of the $\mathcal{O}_\alpha$'s. By the theorem in Appendix B that we cited above there exists a partition of unity, $\widetilde{\rho}_i \in \mathcal{C}_0^\infty(\mathcal{O})$, $i = 1, 2, \ldots$, subordinate to the covering of $\mathcal{O}$ by the $\mathcal{O}_\alpha$'s. Let ρ_i be the restriction of $\widetilde{\rho}_i$ to X. Since the support of $\widetilde{\rho}_i$ is compact and X is closed, the support of ρ_i is compact, so $\rho_i \in \mathcal{C}_0^\infty(X)$ and it's clear that the ρ_i's inherit from the $\widetilde{\rho}_i$'s the properties (a)–(d).

□

Now let the covering (4.5.6) be any covering of X by parametrizable open sets and let $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, \ldots$, be a partition of unity subordinate to this covering. Given $\omega \in \Omega_c^n(X)$ we will define the integral of ω over W by the sum

$$\sum_{i=1}^{\infty} \int_W \rho_i \omega\,. \tag{4.5.8}$$

Note that since each ρ_i is supported in some U_α the individual summands in this sum are well-defined and since the support of ω is compact all but finitely many of these summands are zero by part (b) of Theorem 4.5.1. Hence the sum itself is well-defined. Let's show that this sum doesn't depend on the choice of $\mathbb{U}$ and the ρ_i's. Let $\mathbb{U}'$ be another covering of X by parametrizable open sets and ρ'_j, $j = 1, 2, \dots$, a partition of unity subordinate to $\mathbb{U}'$. Then

$$\begin{aligned} \sum_j \int_W \rho'_j \omega &= \sum_j \int_W \sum_i \rho'_j \rho_i \omega \\ &= \sum_j \left(\sum_i \int_W \rho'_j \rho_i \omega \right) \end{aligned} \tag{4.5.9}$$

by (4.5.4). Interchanging the orders of summation and resuming with respect to the j's this sum becomes

$$\sum_i \int_W \sum_j \rho'_j \rho_i \omega$$

or

$$\sum_i \int_W \rho_i \omega \, .$$

Hence

$$\sum_i \int_W \rho'_j \omega = \sum_i \int_W \rho_i \omega \, ,$$

so the two sums are the same. □

From (4.5.8) and (4.5.4) one easily deduces

Proposition 4.5.3. *For $\omega_i \in \Omega^n_c(X)$, $i = 1, 2$*

$$\int_W \omega_1 + \omega_2 = \int_W \omega_1 + \int_W \omega_2 \tag{4.5.10}$$

and for $\omega \in \Omega^n_c(X)$ and $c \in \mathbb{R}$

$$\int_W c\omega = c \int_W \omega \, . \tag{4.5.11}$$

The definition of the integral (4.5.1) depends on the choice of an orientation of X, but it's easy to see *how* it depends on this choice. We pointed out in Section 4.4 that if X is connected, there is just one way to orient it smoothly other than by its given orientation, namely by *reversing* the orientation of T_p at each point, p, and it's clear from the definitions (4.5.2) and (4.5.8) that the effect of doing this is to change the sign of the integral, i.e., to change $\int_X \omega$ to $-\int_X \omega$.

In the definition of the integral (4.5.1) we've allowed W to be an arbitrary open subset of X but required ω to be compactly supported. This integral is also well-defined if we allow ω to be an arbitrary element of $\Omega^n(X)$ but require the closure of W in X to be compact. To see this, note that under this assumption the sum (4.5.7) is still a finite sum, so the definition of the integral still makes sense, and the double sum on the right side of (4.5.9) is still a finite sum so it's still true that the definition of the integral doesn't depend on the choice of partitions of unity. In particular if the closure of W in X is compact we will define the volume of W to be the integral,

$$\text{vol}(W) = \int_W \sigma_{\text{vol}}\,, \tag{4.5.12}$$

where σ_{vol} is the Riemannian volume form and if X itself is compact we'll define its volume to be the integral

$$\text{vol}(X) = \int_X \sigma_{\text{vol}}\,. \tag{4.5.13}$$

We'll next prove a manifold version of the change of variables formula (3.5.1).

Theorem 4.5.4. *Let X' and X be oriented n-dimensional manifolds and $f : X' \to X$ an orientation preserving diffeomorphism. If W is an open subset of X and $W' = f^{-1}(W)$*

$$\int_{W'} f^*\omega = \int_W \omega \tag{4.5.14}$$

for all $\omega \in \Omega_c^n(X)$.

Proof. By (4.5.8) the integrand of the integral above is a finite sum of $\mathcal{C}^\infty$ forms, each of which is supported on a parametrizable open subset, so we can assume that ω itself as this property. Let V be a parametrizable open set containing the support of ω and let $\varphi_0 : U \to V$ be an oriented parameterization of V. Since f is a diffeomorphism its inverse exists and is a diffeomorphism of X onto X_1. Let $V' = f^{-1}(V)$ and $\varphi_0' = f^{-1} \circ \varphi_0$. Then $\varphi_0' : U \to V'$ is an oriented parameterization of V'. Moreover, $f \circ \varphi_0' = \varphi_0$ so if $W_0 = \varphi_0^{-1}(W)$ we have

$$W_0 = (\varphi_0')^{-1}(f^{-1}(W)) = (\varphi_0')^{-1}(W')$$

and by the chain rule we have

$$\varphi_0^*\omega = (f\circ\varphi_0')^*\omega = (\varphi_0')^* f^*\omega$$

hence

$$\int_W \omega = \int_{W_0} \varphi_0^*\omega = \int_{W_0} (\varphi_0')^*(f^*\omega) = \int_{W'} f^*\omega\,.$$

□

Exercise.

Show that if $f : X' \to X$ is orientation reversing

$$\int_{W'} f^*\omega = -\int_W \omega\,. \tag{4.5.15}$$

We'll conclude this discussion of "integral calculus on manifolds" by proving a preliminary version of Stokes theorem.

Theorem 4.5.5. *If μ is in $\Omega_c^{n-1}(X)$ then*

$$\int_X d\mu = 0\,. \tag{4.5.16}$$

Proof. Let ρ_i, $i = 1, 2, \dots$ be a partition of unity with the property that each ρ_i is supported in a parametrizable open set $U_i = U$. Replacing μ by $\rho_i\mu$ it suffices to prove the theorem for $\mu \in \Omega_c^{n-1}(U)$. Let $\varphi : U_0 \to U$ be an oriented parametrization of U. Then

$$\int_U d\mu = \int_{U_0} \varphi^*\,d\mu = \int_{U_0} d\varphi^*\mu = 0$$

by Theorem 3.3.1.

□

Exercises.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let

$$X = \{(x, x_{n+1}) \in \mathbb{R}^{n+1}\,, \quad x_{n+1} = f(x)\}$$

be the graph of f. Let's orient X by requiring that the diffeomorphism

$$\varphi : \mathbb{R}^n \to X\,, \quad x \to (x, f(x))$$

be orientation preserving. Given a bounded open set U in $\mathbb{R}^n$ compute the Riemannian volume of the image

$$X_U = \varphi(U)$$

of U in X as an integral over U. *Hint:* §4.5, exercise 5.

2. Evaluate this integral for the open subset, X_U, of the paraboloid, $x_3 = x_1^2 + x_2^2$, U being the disk $x_1^2 + x_2^2 < 2$.

3. In exercise 1 let $\iota : X \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map of X onto $\mathbb{R}^{n+1}$.

(a) If $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form, $x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n$, what is the integral of $\iota^*\omega$ over the set X_U? Express this integral as an integral over U.

(b) Same question for $\omega = x_{n+1}^2\, dx_1 \wedge \cdots \wedge dx_n$.

(c) Same question for $\omega = dx_1 \wedge \cdots \wedge dx_n$.

4. Let $f : \mathbb{R}^n \to (0, +\infty)$ be a positive $\mathcal{C}^\infty$ function, U a bounded open subset of $\mathbb{R}^n$, and W the open set of $\mathbb{R}^{n+1}$ defined by the inequalities

$$0 < x_{n+1} < f(x_1, \dots, x_n)$$

and the condition $(x_1, \dots, x_n) \in U$.

(a) Express the integral of the $(n+1)$-form $\omega = x_{n+1}\, dx_1 \wedge \cdots \wedge dx_{n+1}$ over W as an integral over U.

(b) Same question for $\omega = x_{n+1}^2\, dx_1 \wedge \cdots \wedge dx_{n+1}$.

(c) Same question for $\omega = dx_1 \wedge \cdots \wedge dx_n$

5. Integrate the "Riemannian area" form

$$x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2$$

over the unit 2-sphere S^2. (See §4.5, exercise 8.)

Hint: An easier problem: Using polar coordinates integrate $\omega = x_3\, dx_1 \wedge dx_2$ over the hemisphere, $x_3 = \sqrt{1 - x_1^2 - x_2^2}$, $x_1^2 + x_2^2 < 1$.

6. Let α be the one-form $\sum_{i=1}^n y_i\, dx_i$ in formula (2.7.2) and let $\gamma(t)$, $0 \le t \le 1$, be a trajectory of the Hamiltonian vector field (2.7.3). What is the integral of α over $\gamma(t)$?

4.6 Stokes theorem and the divergence theorem

Let X be an oriented n-dimensional manifold and $D \subseteq X$ a smooth domain. We showed in §4.4 that if Z is the boundary of D it acquires from D a natural orientation. Hence if $\iota : Z \to X$ is the inclusion map and μ is in $\Omega_c^{n-1}(X)$, the integral

$$\int_Z \iota^* \mu$$

is well-defined. We will prove:

Theorem 4.6.1 (Stokes theorem)**.** *For* $\mu \in \Omega_c^{k-1}(X)$

$$\int_Z \iota^* \mu = \int_D d\mu \,. \tag{4.6.1}$$

Proof. Let ρ_i, $i = 1, 2, \ldots$, be a partition of unity such that for each i, the support of ρ_i is contained in a parametrizable open set, $U_i = U$, of one of the following three types:

(a) $U \subseteq \operatorname{Int} D$.

(b) $U \subseteq \operatorname{Ext} D$.

(c) There exists an open subset, U_0, of $\mathbb{R}^n$ and an oriented D-adapted parametrization

$$\varphi : U_0 \to U \,. \tag{4.6.2}$$

Replacing μ by the finite sum $\sum \rho_i \mu$ it suffices to prove (4.6.1) for each $\rho_i \mu$ separately. In other words we can assume that the support of μ itself is contained in a parametrizable open set, U, of type (a), (b) or (c). But if U is of type (a)

$$\int_D d\mu = \int_U d\mu = \int_X d\mu$$

and $\iota^* \mu = 0$. Hence the left hand side of (4.6.1) is zero and, by Theorem 4.5.5, the right hand side is as well. If U is of type (b) the situation is even simpler: $\iota^* \mu$ is zero and the restriction of μ to D is zero, so both sides of (4.6.1) are automatically zero. Thus one is reduced to proving (4.6.1) when U is an open subset of type (c).

In this case the restriction of the map (4.6.1) to $U_0 \cap Bd\mathbb{H}^n$ is an orientation preserving diffeomorphism

$$\psi : U_0 \cap Bd\mathbb{H}^n \to U \cap Z \tag{4.6.3}$$

and

$$\iota_Z \circ \psi = \varphi \circ \iota_{\mathbb{R}^{n-1}} \tag{4.6.4}$$

where the maps $\iota = \iota_Z$ and

$$\iota_{\mathbb{R}^{n-1}} : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

are the inclusion maps of Z into X and $Bd\mathbb{H}^n$ into $\mathbb{R}^n$. (Here we're identifying $Bd\mathbb{H}^n$ with $\mathbb{R}^{n-1}$.) Thus

$$\int_D d\mu = \int_{\mathbb{H}^n} \varphi^* \, d\mu = \int_{\mathbb{H}^n} d\varphi^* \mu$$

and by (4.6.4)

$$\begin{aligned} \int_Z \iota_Z^* \mu &= \int_{\mathbb{R}^{n-1}} \psi^* \iota_Z^* \mu \\ &= \int_{\mathbb{R}^{n-1}} \iota_{\mathbb{R}^{n-1}}^* \varphi^* \mu \\ &= \int_{Bd\mathbb{H}^n} \iota_{\mathbb{R}^{n-1}}^* \varphi^* \mu \, . \end{aligned}$$

Thus it suffices to prove Stokes theorem with μ replaced by $\varphi^*\mu$, or, in other words, to prove Stokes theorem for $\mathbb{H}^n$; and this we will now do.

Stokes theorem for $\mathbb{H}^n$: Let

$$\mu = \sum (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \, .$$

Then

$$d\mu = \sum \frac{\partial f_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n$$

and

$$\int_{\mathbb{H}^n} d\mu = \sum_i \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i} \, dx_1 \cdots dx_n \, .$$

We will compute each of these summands as an iterated integral doing the integration with respect to dx_i first. For $i > 1$ the dx_i integration ranges over the interval, $-\infty < x_i < \infty$ and hence since f_i is compactly supported

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}\, dx_i = f_i(x_1, \dots, x_i, \dots, x_n)\Big|_{x_i=-\infty}^{x_i=+\infty} = 0\,.$$

On the other hand the dx_1 integration ranges over the integral, $-\infty < x_1 < 0$ and

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}\, dx_1 = f(0, x_2, \dots, x_n)\,.$$

Thus integrating with respect to the remaining variables we get

$$\int_{\mathbb{H}^n} d\mu = \int_{\mathbb{R}^{n-1}} f(0, x_2, \dots, x_n)\, dx_2 \dots dx_n\,. \tag{4.6.5}$$

On the other hand, since $\iota^*_{\mathbb{R}^{n-1}} x_1 = 0$ and $\iota^*_{\mathbb{R}^{n-1}}\, x_i = x_i$ for $i > 1$,

$$\iota^*_{\mathbb{R}^{n-1}} \mu = f_1(0, x_2, \dots, x_n)\, dx_2 \wedge \cdots \wedge dx_n$$

so

$$\int \iota^*_{\mathbb{R}^{n-1}} \mu = \int f(0, x_2, \dots, x_n)\, dx_2 \dots dx_n\,. \tag{4.6.6}$$

Hence the two sides, (4.6.5) and (4.6.6), of Stokes theorem are equal. □

One important variant of Stokes theorem is the divergence theorem: Let ω be in $\Omega^n_c(X)$ and let v be a vector field on X. Then

$$L_v \omega = \iota(v)\, d\omega + d\iota(v)\omega = d\iota(v)\omega\,,$$

hence, denoting by ι_Z the inclusion map of Z into X we get from Stokes theorem, with $\mu = \iota(v)\omega$:

Theorem 4.6.2 (The manifold version of the divergence theorem)**.**

$$\int_D L_v \omega = \int_Z \iota^*_Z(\iota(v)\omega)\,. \tag{4.6.7}$$

If D is an open domain in $\mathbb{R}^n$ this reduces to the usual divergence theorem of multi-variable calculus. Namely if $\omega = dx_1 \wedge \cdots \wedge dx_n$ and $v = \sum v_i \frac{\partial}{\partial x_i}$ then by (2.4.14)

$$L_v \, dx_1 \wedge \cdots \wedge dx_n = \operatorname{div}(v) \, dx_1 \wedge \cdots \wedge dx_n$$

where

$$\operatorname{div}(v) = \sum \frac{\partial v_i}{\partial x_i} . \tag{4.6.8}$$

Thus if Z is the boundary of D and ι_Z the inclusion map of Z into $\mathbb{R}^n$

$$\int_D \operatorname{div}(v) \, dx = \int_Z \iota_Z^* (\iota_v \, dx_1 \wedge \cdots \wedge dx_n) . \tag{4.6.9}$$

The right hand side of this identity can be interpreted as the "flux" of the vector field, v, through the boundary of D. To see this let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ defining function for D, i.e., a function with the properties

$$p \in D \Leftrightarrow f(p) < 0 \tag{4.6.10}$$

and

$$df_p \neq 0 \text{ if } p \in BdD . \tag{4.6.11}$$

This second condition says that zero is a regular value of f and hence that $Z = BdD$ is defined by the non-degenerate equation:

$$p \in Z \Leftrightarrow f(p) = 0 .$$

Let w be the vector field

$$\left(\sum \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{-1} \sum \frac{\partial f_i}{\partial x_i} \frac{\partial}{\partial x_i} .$$

In view of (4.6.11) this vector field is well-defined on a neighborhood, U, of Z and satisfies

$$\iota(w) \, df = 1 . \tag{4.6.12}$$

Now note that since $df \wedge dx_1 \wedge \cdots \wedge dx_n = 0$

$$\begin{aligned} 0 &= \iota(w)(df \wedge dx_1 \wedge \cdots \wedge dx_n) \\ &= (\iota(w)\, df)\, dx_1 \wedge \cdots \wedge dx_n - df \wedge \iota(w)\, dx_1 \wedge \cdots \wedge dx_n \\ &= dx_1 \wedge \cdots \wedge dx_n - df \wedge \iota(w)\, dx_1 \wedge \cdots \wedge dx_n\,, \end{aligned}$$

hence letting ν be the $(n-1)$-form $\iota(w)\, dx_1 \wedge \cdots \wedge dx_n$ we get the identity

$$dx_1 \wedge \cdots \wedge dx_n = df \wedge \nu \tag{4.6.13}$$

and by applying the operation, $\iota(v)$, to both sides of (4.6.13) the identity

$$\iota(v)\, dx_1 \wedge \cdots \wedge dx_n = (L_v f)\nu - df \wedge \iota(v)\nu\,. \tag{4.6.14}$$

Let $\nu_Z = \iota_Z^* \nu$ be the restriction of ν to Z. Since $\iota_Z^* = 0$, $\iota_Z^*\, df = 0$ and hence by (4.6.14)

$$\iota_Z^*(\iota(v)\, dx_1 \wedge \cdots \wedge dx_n) = \iota_Z^*(L_v f)\nu_Z\,,$$

and the formula (4.6.9) now takes the form

$$\int_D \operatorname{div}(v)\, dx = \int_Z L_v f \nu_Z \tag{4.6.15}$$

where the term on the right is by definition the flux of v through Z. In calculus books this is written in a slightly different form. Letting

$$\sigma_Z = \left(\sum \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{\frac{1}{2}} \nu_Z$$

and letting

$$\vec{n} = \left(\sum \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

and

$$\vec{v} = (v_1, \ldots, v_n)$$

we have

$$L_v \nu_Z = (\vec{n} \cdot \vec{v}) \sigma_Z$$

and hence

$$\int_D \operatorname{div}(v)\, dx = \int_Z (\vec{n} \cdot \vec{v}) \sigma_Z \,. \tag{4.6.16}$$

In three dimensions σ_Z is just the standard "infinitesimal element of area" on the surface Z and n_p the unit outward normal to Z at p, so this version of the divergence theorem is the version one finds in most calculus books.

As an application of Stokes theorem, we'll give a very short alternative proof of the Brouwer fixed point theorem. As we explained in §3.6 the proof of this theorem basically comes down to proving

Theorem 4.6.3. *Let B^n be the closed unit ball in $\mathbb{R}^n$ and S^{n-1} its boundary. Then the identity map*

$$\mathrm{id}_{S^{n-1}} : S^{n-1} \to S^{n-1}$$

can't be extended to a $\mathcal{C}^\infty$ map

$$f : B^n \to S^{n-1} \,.$$

Proof. Suppose that f is such a map. Then for every $n-1$-form, $\mu \in \Omega^{n-1}(S^{n-1})$,

$$\int_{B^n} df^* \mu = \int_{S^{n-1}} (\iota_{S^{n-1}})^* f^* \mu \,. \tag{4.6.17}$$

But $df^*\mu = f^*\, d\mu = 0$ since μ is an $(n-1)$-form and S^{n-1} is an $(n-1)$-dimensional manifold, and since f is the identity map on S^{n-1}, $(\iota_{S_{n-1}})^* f^* \mu = (f \circ \iota_{S^{n-1}})^* \mu = \mu$. Thus for *every* $\mu \in \Omega^{n-1}(S^{n-1})$, (4.6.17) says that the integral of μ over S^{n-1} is zero. Since there are lots of $(n-1)$-forms for which this is not true, this shows that a mapping, f, with the property above can't exist.

□

Exercises.

1. Let B^n be the open unit ball in $\mathbb{R}^n$ and S^{n-1} the unit $(n-1)$-sphere, Show that volume $(S^{n-1}) = n\,\mathrm{volume}\,(B^n)$. *Hint:* Apply

Stokes theorem to the $(n-1)$-form $\mu = \sum (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and note (§4.4, exercise 9) that μ is the Riemannian volume form of S^{n-1}.

2. Let $D \subseteq \mathbb{R}^n$ be a smooth domain with boundary Z. Show that there exists a neighborhood, U, of Z in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ defining function, $g : U \to \mathbb{R}$ for D with the properties

(I) $p \in U \cap D \Leftrightarrow g(p) < 0$.

and

(II) $dg_p \neq 0$ if $p \in Z$

Hint: Deduce from Theorem **??** that a local version of this result is true. Show that you can cover Z by a family

$$\mathbb{U} = \{U_\alpha \, , \, \alpha \in \mathcal{I}\}$$

of open subsets of $\mathbb{R}^n$ such that for each there exists a function, $g_\alpha : U_\alpha \to \mathbb{R}$, with properties (I) and (II). Now let ρ_i, $i = 1, 2, \ldots$, be a partition of unity and let $g = \sum \rho_i g_{\alpha_i}$ where $\operatorname{supp} \rho_i \subseteq U_{\alpha_i}$.

3. In exercise 2 suppose Z is compact. Show that there exists a global defining function, $f : \mathbb{R}^n \to \mathbb{R}$ for D with properties (I) and (II). *Hint:* Let $\rho \in \mathcal{C}_0^\infty(U)$, $0 \leq \rho \leq 1$, be a function which is one on a neighborhood of Z, and replace g by the function

$$f = \begin{cases} \rho g + (1-\rho) \, \text{on} \operatorname{ext} D \\ g \, \text{on} \, Z \\ \rho - g(1-\rho) \, \text{on} \operatorname{int} D \, . \end{cases}$$

4. Show that the form $L_v f \nu_Z$ in formula (4.6.15) doesn't depend on what choice we make of a defining function, f, for D. *Hints:*

(a) Show that if g is another defining function then, at $p \in Z$, $df_p = \lambda \, dg_p$, where λ is a positive constant.

(b) Show that if one replaces df_p by $(dg)_p$ the first term in the product, $(L_v f)(p)(\nu_Z)_p$ changes by a factor, λ, and the second term by a factor $1/\lambda$.

5. Show that the form, ν_Z, is *intrinsically defined* in the sense that if ν is *any* $(n-1)$-form satisfying (4.6.13), ν_Z is equal to $\iota_Z^* \nu$. *Hint:* §4.5, exercise 7.

6. Show that the form, σ_Z, in the formula (4.6.16) is the Riemannian volume form on Z.

7. Show that the $(n-1)$-form

$$\mu = (x_1^2 + \cdots + x_n^2)^{-n} \sum (-1)^{r-1} x_r \, dx_1 \wedge \cdots \wedge \widehat{dx_r} \cdots dx_n$$

is closed and prove directly that Stokes theorem holds for the annulus $a < x_1^2 + \cdots + x_n^2 < b$ by showing that the integral of μ over the sphere, $x_1^2 + \cdots + x_n^2 = a$, is equal to the integral over the sphere, $x_1^2 + \cdots + x_n^2 = b$.

8. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be an everywhere positive $\mathcal{C}^\infty$ function and let U be a bounded open subset of $\mathbb{R}^{n-1}$. Verify directly that Stokes theorem is true if D is the domain

$$0 < x_n < f(x_1, \ldots, x_{n-1}) \,, \quad (x_1, \ldots, x_{n-1}) \in U$$

and μ an $(n-1)$-form of the form

$$\varphi(x_1, \ldots, x_n) \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

where φ is in $\mathcal{C}_0^\infty(\mathbb{R}^n)$.

9. Let X be an oriented n-dimensional manifold and v a vector field on X which is complete. Verify that for $\omega \in \Omega_c^n(X)$

$$\int_X L_v \omega = 0 \,,$$

(a) directly by using the divergence theorem,

(b) indirectly by showing that

$$\int_X f_t^* \omega = \int_X \omega$$

where $f_t : X \to X$, $-\infty < t < \infty$, is the one-parameter group of diffeomorphisms of X generated by v.

10. Let X be an oriented n-dimensional manifold and $D \subseteq X$ a smooth domain whose closure is compact. Show that if Z is the boundary of D and $g : Z \to Z$ a diffeomorphism, g *can't* be extended to a smooth map, $f : D \to Z$.

4.7 Degree theory on manifolds

In this section we'll show how to generalize to manifolds the results about the "degree" of a proper mapping that we discussed in Chapter 3. We'll begin by proving the manifold analogue of Theorem 3.3.1.

Theorem 4.7.1. *Let X be an oriented connected n-dimensional manifold and $\omega \in \Omega_c^n(X)$ a compactly supported n-form. Then the following are equivalent*

(a) $\int_X \omega = 0$.

(b) $\omega = d\mu$ *for some* $\mu \in \Omega_c^{n-1}(X)$.

We've already verified the assertion (b) $\Rightarrow$ (a) (see Theorem **??**), so what is left to prove is the converse assertion. The proof of this is more or less identical with the proof of the "(a) $\Rightarrow$ (b)" part of Theorem 3.2.1:

Step 1. Let U be a connected parametrizable open subset of X. If $\omega \in \Omega_c^n(U)$ has property (a), then $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(U)$.

Proof. Let $\varphi : U_0 \to U$ be an oriented parametrization of U. Then

$$\int_{U_0} \varphi^* \omega = \int_X \omega = 0$$

and since U_0 is a connected open subset of $\mathbb{R}^n$, $\varphi^*\omega = d\nu$ for some $\nu \in \Omega_c^{n-1}(U_0)$ by Theorem 3.3.1. Let $\mu = (\varphi^{-1})^*\nu$. Then $d\mu = (\varphi^{-1})^* \, d\nu = \omega$.

□

Step 2. Fix a base point, $p_0 \in X$ and let p be any point of X. Then there exists a collection of connected parametrizable open sets, W_i, $i = 1, \ldots, N$ with $p_0 \in W_1$ and $p \in W_N$ such that, for $1 \le i \le N-1$, the intersection of W_i and W_{i+1} is non-empty.

Proof. The set of points, $p \in X$, for which this assertion is true is open and the set for which it is not true is open. Moreover, this assertion is true for $p = p_0$.

□

Step 3. We deduce Theorem 4.7.1 from a slightly stronger result. Introduce an equivalence relation on $\Omega_c^n(X)$ by declaring that two n-forms, ω_1 and ω_2, in $\Omega_c^n(X)$ are *equivalent* if $\omega_1 - \omega_2 \in d\Omega_x^{n-1}(X)$. Denote this equivalence relation by a wiggly arrow: $\omega_1 \sim \omega_2$. We will prove

Theorem 4.7.2. *For ω_1 and $\omega_2 \in \Omega_c^n(X)$ the following are equivalent*

(a) $\int_X \omega_1 = \int_X \omega_2$

(b) $\omega_1 \sim \omega_2$.

Applying this result to a form, $\omega \in \Omega_c^n(X)$, whose integral is zero, we conclude that $\omega \sim 0$, which means that $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(X)$. Hence Theorem 4.7.2 implies Theorem 4.7.1. Conversely, if $\int_X \omega_1 = \int_X \omega_2$. Then $\int_X (\omega_1 - \omega_2) = 0$, so $\omega_1 - \omega_2 = d\mu$ for some $\mu \in \Omega_c^n(X)$. Hence Theorem 4.7.1 implies Theorem 4.7.2.

Step 4. By a partition of unity argument it suffices to prove Theorem 4.7.2 for $\omega_1 \in \Omega_c^n(U_1)$ and $\omega_2 \in \Omega_c^n(U_2)$ where U_1 and U_2 are connected parametrizable open sets. Moreover, if the integrals of ω_1 and ω_2 are zero then $\omega_i = d\mu_i$ for some $\mu_i \in \Omega_c^n(U_i)$ by step 1, so in this case, the theorem is true. Suppose on the other hand that

$$\int_X \omega_1 = \int_X \omega_2 = c \neq 0\,.$$

Then dividing by c, we can assume that the integrals of ω_1 and ω_2 are both equal to 1.

Step 5. Let W_i, $i = 1, \dots, N$ be, as in step 2, a sequence of connected parametrizable open sets with the property that the intersections, $W_1 \cap U_1$, $W_N \cap U_2$ and $W_i \cap W_{i+1}$, $i = 1, \dots, N-1$, are all non-empty. Select n-forms, $\alpha_0 \in \Omega_c^n(U_1 \cap W_1)$, $\alpha_N \in \Omega_c^n(W_N \cap U_2)$ and $\alpha_i \in \Omega_c^n(W_i \cap W_{i+1})$, $i = 1, \dots, N-1$ such that the integral of each α_i over X is equal to 1. By step 1 Theorem 4.7.1 is true for U_1, U_2 and the W_i's, hence Theorem 4.7.2 is true for U_1, U_2 and the W_i's, so

$$\omega_1 \sim \alpha_0 \sim \alpha_1 \sim \cdots \sim \alpha_N \sim \omega_2$$

and thus $\omega_1 \sim \omega_2$. □

Just as in (3.4.1) we get as a corollary of the theorem above the following "definition–theorem" of the degree of a differentiable mapping:

Theorem 4.7.3. *Let X and Y be compact oriented n-dimensional manifolds and let Y be connected. Given a proper $\mathcal{C}^\infty$ mapping, $f : X \to Y$, there exists a topological invariant,* $\deg(f)$, *with the defining property:*

$$\int_X f^*\omega = \deg f \int_Y \omega \,. \tag{4.7.1}$$

Proof. As in the proof of Theorem 3.4.1 pick an n-form, $\omega_0 \in \Omega_c^n(Y)$, whose integral over Y is one and define the degree of f to be the integral over X of $f^*\omega_0$, i.e., set

$$\deg(f) = \int_X f^*\omega_0 \,. \tag{4.7.2}$$

Now let ω be any n-form in $\Omega_c^n(Y)$ and let

$$\int_Y \omega = c \,. \tag{4.7.3}$$

Then the integral of $\omega - c\omega_0$ over Y is zero so there exists an $(n-1)$-form, μ, in $\Omega_c^{n-1}(Y)$ for which $\omega - c\omega_0 = d\mu$. Hence $f^*\omega = cf^*\omega_0 + df^*\mu$, so

$$\int_X f^*\omega = c\int_X f^*\omega_0 = \deg(f)\int_Y \omega$$

by (4.7.2) and (4.7.3).

□

It's clear from the formula (4.7.1) that the degree of f is independent of the choice of ω_0. (Just apply this formula to any $\omega \in \Omega_c^n(Y)$ having integral over Y equal to one.) It's also clear from (4.7.1) that "degree" behaves well with respect to composition of mappings:

Theorem 4.7.4. *Let Z be an oriented, connected n-dimensional manifold and $g : Y \to Z$ a proper $\mathcal{C}^\infty$ map. Then*

$$\deg g \circ f = (\deg f)(\deg g) \,. \tag{4.7.4}$$

Proof. Let ω be an element of $\Omega^n_c(Z)$ whose integral over Z is one. Then

$$\begin{aligned} \deg g \circ f = \int_X (g \circ f)^* \omega &= \int_X f^* \circ g^* \omega = \deg f \int_Y g^* \omega \\ &= (\deg f)(\deg g)\,. \end{aligned}$$

□

We will next show how to compute the degree of f by generalizing to manifolds the formula for $\deg(f)$ that we derived in §3.6.

Definition 4.7.5. *A point, $p \in X$ is a critical point of f if the map*

$$df_p : T_pX \to T_{f(p)}Y \tag{4.7.5}$$

is not bijective.

We'll denote by C_f the set of all critical points of f, and we'll call a point $q \in Y$ a *critical* value of f if it is in the image, $f(C_f)$, of C_f and a *regular* value if it's not. (Thus the set of regular values is the set, $Y - f(C_f)$.) If q is a regular value, then as we observed in §3.6, the map (4.7.5) is bijective for every $p \in f^{-1}(q)$ and hence by Theorem 4.2.5, f maps a neighborhood U_p of p diffeomorphically onto a neighborhood, V_p, of q. In particular, $U_p \cap f^{-1}(q) = p$. Since f is proper the set $f^{-1}(q)$ is compact, and since the sets, U_p, are a covering of $f^{-1}(q)$, this covering must be a finite covering. In particular the set $f^{-1}(q)$ itself has to be a finite set. As in §2.6 we can shrink the U_p's so as to insure that they have the following properties:

(i) Each U_p is a parametrizable open set.

(ii) $U_p \cap U_{p'}$ is empty for $p \neq p'$.

(iii) $f(U_p) = f(U_{p'}) = V$ for all p and p'.

(iv) V is a parametrizable open set.

(v) $f^{-1}(V) = \bigcup U_p$, $p \in f^{-1}(q)$.

To exploit these properties let ω be an n-form in $\Omega^n_c(V)$ with integral equal to 1. Then by (v):

$$\deg(f) = \int_X f^* \omega = \sum_p \int_{U_p} f^* \omega\,.$$

But $f : U_p \to V$ is a diffeomorphism, hence by (4.5.14) and (4.5.15)

$$\int_{U_p} f^*\omega = \int_V \omega$$

if $f : U_p \to V$ is orientation preserving and

$$\int_{U_p} f^*\omega = -\int_V \omega$$

if $f : U_p \to V$ is orientation reversing. Thus we've proved

Theorem 4.7.6. *The degree of f is equal to the sum*

$$\sum_{p \in f^{-1}(q)} \sigma_p \tag{4.7.6}$$

where $\sigma_p = +1$ if the map (4.7.5) is orientation preserving and $\sigma_p = -1$ if it is orientation reversing.

We will next show that Sard's Theorem is true for maps between manifolds and hence that there exist lots of regular values. We first observe that if U is a parametrizable open subset of X and V a parametrizable open neighborhood of $f(U)$ in Y, then Sard's Theorem is true for the map, $f : U \to V$ since, up to diffeomorphism, U and V are just open subsets of $\mathbb{R}^n$. Now let q be any point in Y, let B be a compact neighborhood of q, and let V be a parametrizable open set containing B. Then if $A = f^{-1}(B)$ it follows from Theorem 3.4.2 that A can be covered by a finite collection of parametrizable open sets, $U_1, \ldots, U_N$ such that $f(U_i) \subseteq V$. Hence since Sard's Theorem is true for each of the maps $f : U_i \to V$ and $f^{-1}(B)$ is contained in the union of the U_i's we conclude that *the set of regular values of f intersects the interior of B in an open dense set.* Thus, since q is an arbitrary point of Y, we've proved

Theorem 4.7.7. *If X and Y are n-dimensional manifolds and $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map the set of regular values of f is an open dense subset of Y.*

Since there exist lots of regular values the formula (4.7.6) gives us an effective way of computing the degree of f. We'll next justify our assertion that $\deg(f)$ is a topological invariant of f. To do so, let's generalize to manifolds the Definition 2.5.1, of a homotopy between $\mathcal{C}^\infty$ maps.

Definition 4.7.8. *Let X and Y be manifolds and $f_i : X \to Y$, $i = 0, 1$, a $\mathcal{C}^\infty$ map. A $\mathcal{C}^\infty$ map*

$$F : X \times [0,1] \to Y \tag{4.7.7}$$

is a homotopy between f_0 and f_1 if, for all $x \in X$, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Moreover, if f_0 and f_1 are proper maps, the homotopy, F, is a proper *homotopy if it is proper as a $\mathcal{C}^\infty$ map, i.e., for every compact set, C, of Y, $F^{-1}(C)$ is compact.*

Let's now prove the manifold analogue of Theorem 3.6.8.

Theorem 4.7.9. *Let X and Y be oriented n-dimensional manifolds and let Y be connected. Then if $f_i : X \to Y$, $i = 0, 1,$, is a proper map and the map (4.7.4) is a property homotopy, the degrees of these maps are the same.*

Proof. Let ω be an n-form in $\Omega_c^n(Y)$ whose integral over Y is equal to 1, and let C be the support of ω. Then if F is a proper homotopy between f_0 and f_1, the set, $F^{-1}(C)$, is compact and its projection on X

$$\{x \in X \,;\, (x,t) \in F^{-1}(C) \text{ for some } t \in [0,1]\} \tag{4.7.8}$$

is compact. Let

$$f_t : X \to Y$$

be the map: $f_t(x) = F(x,t)$. By our assumptions on F, f_t is a proper $\mathcal{C}^\infty$ map. Moreover, for all t the n-form, $f_t^*\omega$ is a $\mathcal{C}^\infty$ function of t and is supported on the fixed compact set (4.7.8). Hence it's clear from the Definition 4.6.8 that the integral

$$\int_X f_t^*\omega$$

is a $\mathcal{C}^\infty$ function of t. On the other hand this integral is by definition the degree of f_t and hence by Theorem 4.7.3 is an integer, so it doesn't depend on t. In particular, $\deg(f_0) = \deg(f_1)$. □

Exercises.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the map, $x \to x^n$. Show that $\deg(f) = 0$ if n is even and 1 if n is odd.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial function,

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \,,$$

where the a_i's are in $\mathbb{R}$. Show that if n is even, $\deg(f) = 0$ and if n is odd, $\deg(f) = 1$.

3. Let S^1 be the unit circle

$$\{e^{i\theta}\,, \quad 0 \le \theta < 2\pi\}$$

in the complex plane and let $f : S^1 \to S^1$ be the map, $e^{i\theta} \to e^{iN\theta}$, N being a positive integer. What's the degree of f?

4. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and $\sigma : S^{n-1} \to S^{n-1}$ the antipodal map, $x \to -x$. What's the degree of σ?

5. Let A be an element of the group, $O(n)$ of orthogonal $n \times n$ matrices and let

$$f_A : S^{n-1} \to S^{n-1}$$

be the map, $x \to Ax$. What's the degree of f_A?

6. A manifold, Y, is *contractable* if for some point, $p_0 \in Y$, the identity map of Y onto itself is homotopic to the constant map, $f_{p_0} : Y \to Y$, $f_{p_0}(y) = p_0$. Show that if Y is an oriented contractable n-dimensional manifold and X an oriented connected n-dimensional manifold then for every proper mapping $f : X \to Y$ $\deg(f) = 0$. In particular show that if n is greater than zero and Y is compact then Y can't be contractable. *Hint:* Let f be the identity map of Y onto itself.

7. Let X and Y be oriented connected n-dimensional manifolds and $f : X \to Y$ a proper $\mathcal{C}^\infty$ map. Show that if $\deg(f) \neq 0$ f is surjective.

8. Using Sard's Theorem prove that if X and Y are manifolds of dimension k and ℓ, with $k < \ell$ and $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map, then the complement of the image of X in Y is open and dense. *Hint:* Let $r = \ell - k$ and apply Sard's Theorem to the map

$$g : X \times S^r \to Y\,, \quad g(x,a) = f(x)\,.$$

9. Prove that the sphere, S^2, and the torus, $S^1 \times S^2$, are not diffeomorphic.

4.8 Applications of degree theory

The purpose of this section will be to describe a few typical applications of degree theory to problems in analysis, geometry and topology. The first of these applications will be yet another variant of the Brouwer fixed point theorem.

Application 1. Let X be an oriented $(n+1)$-dimensional manifold, $D \subseteq X$ a smooth domain and Z the boundary of D. Assume that the closure, $\bar{D} = Z \cup D$, of D is compact (and in particular that X is compact).

Theorem 4.8.1. *Let Y be an oriented connected n-dimensional manifold and $f : Z \to Y$ a $\mathcal{C}^\infty$ map. Suppose there exists a $\mathcal{C}^\infty$ map, $F : \bar{D} \to Y$ whose restriction to Z is f. Then the degree of f is zero.*

Proof. Let μ be an element of $\Omega_c^n(Y)$. Then $d\mu = 0$, so $dF^*\mu = F^*\,d\mu = 0$. On the other hand if $\iota : Z \to X$ is the inclusion map,

$$\int_D dF^*\mu = \int_Z \iota^* F^*\mu = \int_Z f^*\mu = \deg(f) \int_Y \mu$$

by Stokes theorem since $F \circ \iota = f$. Hence $\deg(f)$ has to be zero. $\square$

Application 2. (a non-linear eigenvalue problem)

This application is a non-linear generalization of a standard theorem in linear algebra. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If n is even, A may not have *real* eigenvalues. (For instance for the map

$$A : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad (x, y) \to (-y, x)$$

the eigenvalues of A are $\pm\sqrt{-1}$.) However, if n is odd it is a standard linear algebra fact that there exists a vector, $\mathrm{v} \in \mathbb{R}^n - \{0\}$, and a $\lambda \in \mathbb{R}$ such that $A\mathrm{v} = \lambda\mathrm{v}$. Moreover replacing v by $\frac{\mathrm{v}}{|\mathrm{v}|}$ one can assume that $|\mathrm{v}| = 1$. This result turns out to be a special case of a much more general result. Let S^{n-1} be the unit $(n-1)$-sphere in $\mathbb{R}^n$ and let $f : S^{n-1} \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map.

Theorem 4.8.2. *There exists a vector, $\mathrm{v} \in S^{n-1}$ and a number $\lambda \in \mathbb{R}$ such that $f(\mathrm{v}) = \lambda\mathrm{v}$.*

Proof. The proof will be by contradiction. If the theorem isn't true the vectors, v and $f(\mathrm{v})$, are linearly independent and hence the vector

$$g(\mathrm{v}) = f(\mathrm{v}) - (f(\mathrm{v}) \cdot \mathrm{v})\mathrm{v} \tag{4.8.1}$$

is non-zero. Let

$$h(\mathrm{v}) = \frac{g(\mathrm{v})}{|g(\mathrm{v})|} . \tag{4.8.2}$$

By (4.8.1)–(4.8.2), $|\mathrm{v}| = |h(\mathrm{v})| = 1$ and $\mathrm{v} \cdot h(\mathrm{v}) = 0$, i.e., v and $h(\mathrm{v})$ are both unit vectors and are perpendicular to each other. Let

$$\gamma_t : S^{n-1} \to S^{n-1} , \quad 0 \leq t \leq 1 \tag{4.8.3}$$

be the map

$$\gamma_t(\mathrm{v}) = (\cos \pi t)\mathrm{v} + (\sin \pi t)h(\mathrm{v}) . \tag{4.8.4}$$

For $t = 0$ this map is the identity map and for $t = 1$, it is the antipodal map, $\sigma(\mathrm{v}) = \mathrm{v}$, hence (4.8.3) asserts that the identity map and the antipodal map are homotopic and therefore that the degree of the antipodal map is one. On the other hand the antipodal map is the restriction to S^{n-1} of the map, $(x_1, \dots, x_n) \to (-x_1, \dots, -x_n)$ and the volume form, ω, on S^{n-1} is the restriction to S^{n-1} of the $(n-1)$-form

$$\sum (-1)^{i-1} x_i \, dx_i \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n . \tag{4.8.5}$$

If we replace x_i by $-x_i$ in (4.8.5) the sign of this form changes by $(-1)^n$ hence $\sigma^*\omega = (-1)^n\omega$. Thus if n is odd, σ is an orientation reversing diffeomorphism of S^{n-1} onto S^{n-1}, so its degree is -1, and this contradicts what we just deduced from the existence of the homotopy (4.8.4).

□

From this argument we can deduce another interesting fact about the sphere, S^{n-1}, when $n-1$ is even. For $\mathrm{v} \in S^{n-1}$ the tangent space to S^{n-1} at v is just the space,

$$\{(\mathrm{v}, w) ; \quad w \in \mathbb{R}^n , \ \mathrm{v} \cdot w = 0\} ,$$

so a vector field on S^{n-1} can be viewed as a function, $g : S^{n-1} \to \mathbb{R}^n$ with the property

$$g(\mathrm{v}) \cdot \mathrm{v} = 0 \tag{4.8.6}$$

for all $\mathrm{v} \in S^{n-1}$. If this function is non-zero at all points, then, letting h be the function, (4.8.2), and arguing as above, we're led to a contradiction. Hence we conclude:

Theorem 4.8.3. *If $n-1$ is even and v is a vector field on the sphere, S^{n-1}, then there exists a point $p \in S^{n-1}$ at which $v(p) = 0$.*

Note that if $n-1$ is odd this statement is *not* true. The vector field

$$x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \cdots + x_{2n-1} \frac{\partial}{\partial x_{2n}} - x_{2n} \frac{\partial}{\partial x_{2n-1}} \tag{4.8.7}$$

is a counterexample. It is nowhere vanishing and at $p \in S^{n-1}$ is tangent to S^{n-1}.

Application 3. (The Jordan–Brouwer separation theorem.) Let X be a compact oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. In this subsection of §4.8 we'll outline a proof of the following theorem (leaving the details as a string of exercises).

Theorem 4.8.4. *If X is connected, the complement of $X : \mathbb{R}^n - X$ has exactly two connected components.*

This theorem is known as the Jordan-Brouwer separation theorem (and in two dimensions as the Jordan curve theorem). For simple, easy to visualize, submanifolds of $\mathbb{R}^n$ like the $(n-1)$-sphere this result is obvious, and for this reason it's easy to be misled into thinking of it as being a trivial (and not very interesting) result. However, for submanifolds of $\mathbb{R}^n$ like the curve in $\mathbb{R}^2$ depicted in the figure below it's much less obvious. (In ten seconds or less, is the point, p, in this figure inside this curve or outside?)

Figure . Guillemin-Pollack, p. 86 fig. 2-19

To determine whether a point, $p \in \mathbb{R}^n - X$ is inside X or outside X, one needs a topological invariant to detect the difference, and such an invariant is provided by the "winding number".

Definition 4.8.5. *For $p \in \mathbb{R}^n - X$ let*

$$\gamma_p : X \to S^{n-1} \tag{4.8.8}$$

be the map

$$\gamma_p(x) = \frac{x-p}{|x-p|} \,. \tag{4.8.9}$$

The *winding number* of X about p is the degree of this map.

Denoting this number by $W(X,p)$ we will show below that $W(X,p) = 0$ if p is outside X and $W(X,p) = \pm 1$ (depending on the orientation of X) if p is inside X, and hence that the winding number tells us which of the two components of $\mathbb{R}^n - X$, p is contained in.

Exercise 1.

Let U be a connected component of $\mathbb{R}^n - X$. Show that if p_0 and p_1 are in U, $W(X,p_0) = W(X,p_1)$.

Hints:

(a) First suppose that the line segment,

$$p_t = (1-t)p_0 + tp_1 \,, \quad 0 \le t \le 1$$

lies in U. Conclude from the homotopy invariance of degree that $W(X,p_0) = W(X,p_t) = W(X,p_1)$.

(b) Show that there exists a sequence of points

$$q_i \,, \quad i = 1, \ldots, N \,, \quad q_i \in U \,,$$

with $q_1 = p_0$ and $q_N = p_1$, such that the line segment joining q_i to q_{i+1} is in U.

Exercise 2.

Show that $\mathbb{R}^n - X$ has at most two connected components.

Hints:

(a) Show that if q is in X there exists a small ϵ-ball, $B_\epsilon(q)$, centered at q such that $B_\epsilon(q) - X$ has two components. (See Theorem **??**.

(b) Show that if p is in $\mathbb{R}^n - X$, there exists a sequence

$$q_i\,,\ i = 1, \dots, N\,, \quad q_i \in \mathbb{R}^n - X\,,$$

such that $q_1 = p$, $q_N \in B_\epsilon(q)$ and the line segments joining q_i to q_{i+1} are in $\mathbb{R}^n - X$.

Exercise 3.

For $\mathrm{v} \in S^{n-1}$, show that $x \in X$ is in $\gamma_p^{-1}(\mathrm{v})$ if and only if x lies on the ray

$$p + t\mathrm{v}\,, \quad 0 < t < \infty\,. \tag{4.8.10}$$

Exercise 4.

Let $x \in X$ be a point on this ray. Show that

$$(d\gamma_p)_x : T_pX \to T_{\mathrm{v}}S^{n-1} \tag{4.8.11}$$

is bijective if and only if $v \notin T_pX$, i.e., if and only if the ray (4.8.10) is *not* tangent to X at x. *Hint:* $\gamma_p : X \to S^{n-1}$ is the composition of the maps

$$\tau_p : X \to \mathbb{R}^n - \{0\}\,, \quad x \to x - p\,, \tag{4.8.12}$$

and

$$\pi : \mathbb{R}^n - \{0\} \to S^{n-1}\,, \quad y \to \frac{y}{|y|}\,. \tag{4.8.13}$$

Show that if $\pi(y) = \mathrm{v}$, then the kernel of $(d\pi)_g$, is the one-dimensional subspace of $\mathbb{R}^n$ spanned by v. Conclude that if $y = x - p$ and $\mathrm{v} = y/|y|$ the composite map

$$(d\gamma_p)_x = (d\pi)_y \circ (d\tau_p)_x$$

is bijective if and only if $\mathrm{v} \notin T_xX$.

Exercise 5.

From exercises 3 and 4 conclude that v is a regular value of γ_p if and only if the ray (4.8.10) intersects X in a finite number of points and at each point of intersection is *not* tangent to X at that point.

Exercise 6.

In exercise 5 show that the map (4.8.11) is orientation preserving if the orientations of T_xX and v are compatible with the standard orientation of $T_p\mathbb{R}^n$. (See §1.9, exercise 5.)

Exercise 7.

Conclude that $\deg(\gamma_p)$ counts (with orientations) the number of points where the ray (4.8.10) intersects X.

Exercise 8.

Let $p_1 \in \mathbb{R}^n - X$ be a point on the ray (4.8.10). Show that if $\mathrm{v} \in S^{n-1}$ is a regular value of γ_p, it is a regular value of γ_{p_1} and show that the number

$$\deg(\gamma_p) - \deg(\gamma_{p_1}) = W(X,p) - W(X,p_1)$$

counts (with orientations) the number of points on the ray lying between p and p_1. *Hint:* Exercises 5 and 7.

Exercise 8.

Let $x \in X$ be a point on the ray (4.8.10). Suppose $x = p + t\mathrm{v}$. Show that if ϵ is a small positive number and

$$p_\pm = p + (t \pm \epsilon)\mathrm{v}$$

then

$$W(X,p_+) = W(X,p_-) \pm 1\,,$$

and from exercise 1 conclude that p_+ and p_- lie in different components of $\mathbb{R}^n - X$. In particular conclude that $\mathbb{R}^n - X$ has exactly two components.

Exercise 9.

Finally show that if p is very large the difference

$$\gamma_p(x) - \frac{p}{|p|}\,, \quad x \in X\,,$$

is very small, i.e., γ_p is *not* surjective and hence the degree of γ_p is zero. Conclude that for $p \in \mathbb{R}^n - X$, p is in the unbounded component of $\mathbb{R}^n - X$ if $W(X,p) = 0$ and in the bounded component if $W(X,p) = \pm 1$ (the "$\pm$" depending on the orientation of X).

Notice, by the way, that the proof of Jordan–Brouwer sketched above gives us an effective way of deciding whether the point, p, in Figure 4.8, is inside X or outside X. Draw a non-tangential ray from p as in Figure 4.9.2. If it intersects X in an even number of points, p is outside X and if it intersects X is an odd number of points p inside.

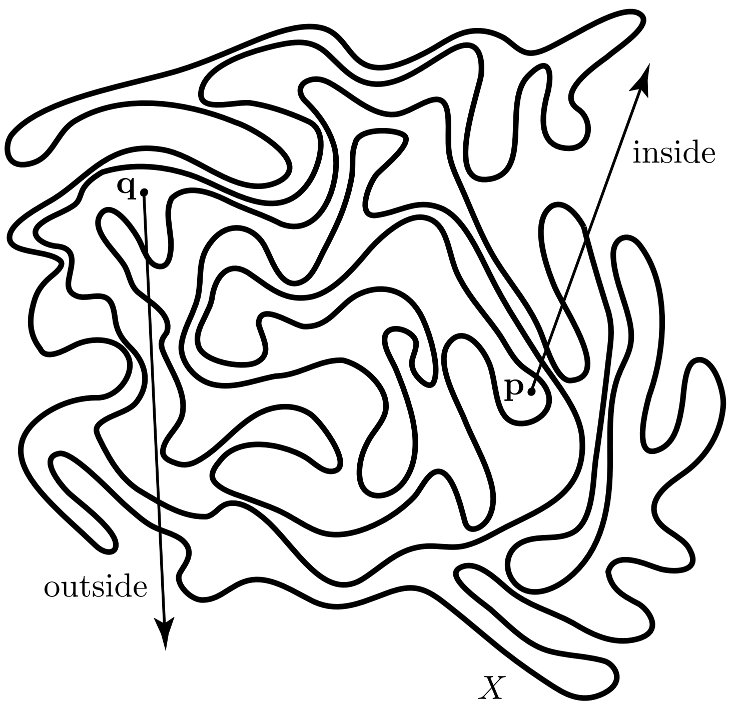


Figure 4.9.2.

Application 3. (The Gauss–Bonnet theorem.) Let $X \subseteq \mathbb{R}^n$ be a compact, connected, oriented $(n-1)$-dimensional submanifold. By the Jordan–Brouwer theorem X is the boundary of a bounded smooth domain, so for each $x \in X$ there exists a unique outward pointing unit normal vector, n_x. The Gauss map

$$\gamma : X \to S^{n-1}$$

is the map, $x \to n_x$. Let σ be the Riemannian volume form of S^{n-1}, or, in other words, the restriction to S^{n-1} of the form,

$$\sum (-1)^{i-1} x_i \, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n \, ,$$

and let σ_X be the Riemannian volume form of X. Then for each $p \in X$

$$(\gamma^* \sigma)_p = K(p)(\sigma_X)_q \tag{4.8.14}$$

where $K(p)$ is the *scalar curvature* of X at p. This number measures the extent to which "X is curved" at p. For instance, if X_a is the circle, $|x| = a$ in $\mathbb{R}^2$, the Gauss map is the map, $p \to p/a$, so for all p, $K_a(p) = 1/a$, reflecting the fact that, for $a < b$, X_a is more curved than X_b.

The scalar curvature can also be negative. For instance for surfaces, X in $\mathbb{R}^3$, $K(p)$ is positive at p if X is *convex* at p and negative if X is convex–concave at p. (See Figure 4.9.3 below. The surface in part (a) is convex at p, and the surface in part (b) is convex–concave.)

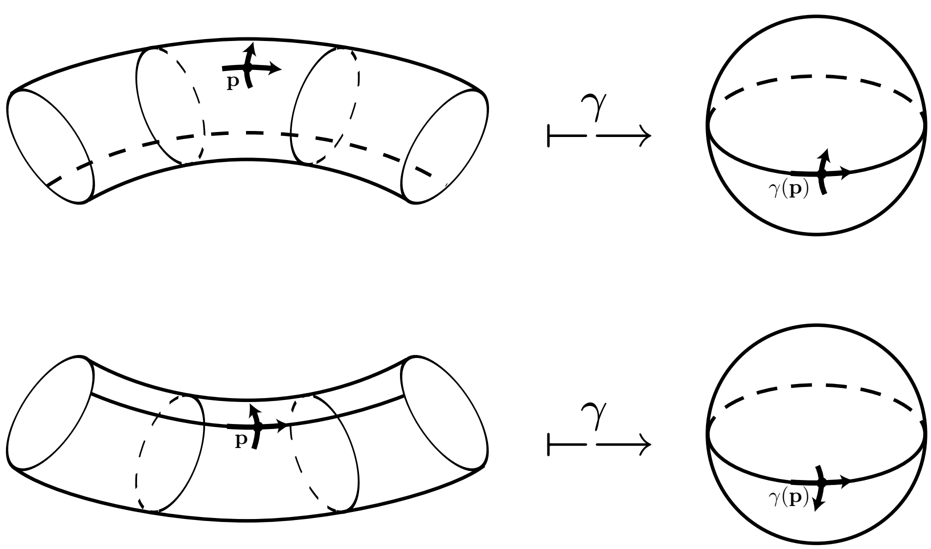


Figure 4.9.3.

Let vol (S^{n-1}) be the Riemannian volume of the $(n-1)$-sphere, i.e., let

$$\mathrm{vol}\,(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(where Γ is the gamma function). Then by (4.8.14) the quotient

$$\frac{\int K\sigma_X}{\mathrm{vol}\,(S^{n-1})} \tag{4.8.15}$$

is the degree of the Gauss map, and hence is a topological invariant of the surface of X. For $n = 3$ the *Gauss–Bonnet theorem* asserts that this topological invariant is just $1 - g$ where g is the *genus* of X or, in other words, the "number of holes". Figure 4.9.4 gives a pictorial proof of this result. (Notice that at the points, $p_1, \dots, p_g$ the surface,X is convex–concave so the scalar curvature at these points is negative, i.e., the Gauss map is orientation reversing. On the other hand, at the point, p_0, the surface is convex, so the Gauss map at this point us orientation preserving.)

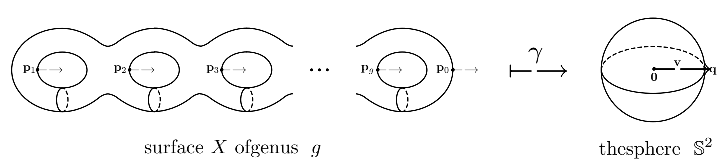


Figure 4.9.4.

CHAPTER 5

COHOMOLOGY VIA FORMS

5.1 The DeRham cohomology groups of a manifold

In the last four chapters we've frequently encountered the question: When is a closed k-form on an open subset of $\mathbb{R}^N$ (or, more generally on a submanifold of $\mathbb{R}^N$) exact? To investigate this question more systematically than we've done heretofore, let X be an n-dimensional manifold and let

(5.1.1) $$Z^k(X) = \{\omega \in \Omega^k(X)\,;\, d\omega = 0\}$$

and

(5.1.2) $$B^k(X) = \{\omega \in \Omega^k(X)\,;\, \omega \text{ in } d\Omega^{k-1}(X)\}$$

be the vector spaces of closed and exact k-forms. Since (5.1.2) is a vector subspace of (5.1.1) we can form the quotient space

(5.1.3) $$H^k(X) = Z^k(X)/B^k(X)\,,$$

and the dimension of this space is a measure of the extent to which closed forms fail to be exact. We will call this space the k^{th} *DeRham cohomology group of the manifold*, X. Since the vector spaces (5.1.1) and (5.1.2) are both infinite dimensional there is no guarantee that this quotient space is finite dimensional, however, we'll show later in this chapter that it is in lots of interesting cases.

The spaces (5.1.3) also have compactly supported counterparts. Namely let

(5.1.4) $$Z^k_c(X) = \{\omega \in \Omega^k_c(X)\,;\, d\omega = 0\}$$

and

(5.1.5) $$B^k_c(X) = \{\omega \in \Omega^k_c(X)\,,\, \omega \text{ in } d\Omega^{k-1}_c(X)\}\,.$$

Then as above $B^k_c(X)$ is a vector subspace of $Z^k_c(X)$ and the vector space quotient

(5.1.6) $$H^k_c(X) = Z^k_c(X)/B^k_c(X)$$

is the k^{th} *compactly supported DeRham cohomology group of* X.

Given a closed k-form, $\omega \in Z^k(X)$, we will denote by $[\omega]$ the image of ω in the quotient space (5.1.3) and call $[\omega]$ the *cohomology class* of ω. We will also use the same notation for compactly supported cohomology. If ω is in $Z^k_c(X)$ we'll denote by $[\omega]$ the cohomology class of ω in the quotient space (5.1.6).

Some cohomology groups of manifolds we've already computed in the previous chapters (although we didn't explicitly describe these computations as "computing cohomology"). We'll make a list below of some of the things we've already learned about DeRham cohomology:

1. If X is connected, $H^0(X) = \mathbb{R}$. Proof: A closed zero form is a function, $f \in \mathcal{C}^\infty(X)$ having the property, $df = 0$, and if X is connected the only such functions are constants.

2. If X is connected and non-compact $H^0_c(X) = \{0\}$. Proof: If f is in $\mathcal{C}^\infty_0(X)$ and X is non-compact, f has to be zero at some point, and hence if $df = 0$ it has to be identically zero.

3. If X is n-dimensional,

$$\Omega^k(X) = \Omega^k_c(X) = \{0\}$$

for k less than zero or k greater than n, hence

$$H^k(X) = H^k_c(X) = \{0\}$$

for k less than zero or k greater than n.

4. If X is an oriented, connected n-dimensional manifold, the integration operation is a linear map

$$\int : \Omega^n_c(X) \to \mathbb{R} \tag{5.1.7}$$

and, by Theorem 4.8.1, the kernel of this map is $B^n_c(X)$. Moreover, in degree n, $Z^n_c(X) = \Omega^n_c(X)$ and hence by (5.1.6), we get from (5.1.7) a bijective map

$$I_X : H^n_c(X) \to \mathbb{R}\,. \tag{5.1.8}$$

In other words

$$H^n_c(X) = \mathbb{R}\,. \tag{5.1.9}$$

5. Let U be a star-shaped open subset of $\mathbb{R}^n$. In §2.5, exercises 4–7, we sketched a proof of the assertion: For $k > 0$ every closed form, $\omega \in Z^k(U)$ is exact, i.e., translating this assertion into cohomology language, we showed that

$$H^k(U) = \{0\} \text{ for } k > 0\,. \tag{5.1.10}$$

6. Let $U \subseteq \mathbb{R}^n$ be an open rectangle. In §3.2, exercises 4–7, we sketched a proof of the assertion: If $\omega \in \Omega_c^k(U)$ is closed and k is less than n, then $\omega = d\mu$ for some $(k-1)$-form, $\mu \in \Omega_c^{k-1}(U)$. Hence we showed

$$H_c^k(U) = 0 \text{ for } k < n\,. \tag{5.1.11}$$

7. *Poincaré's lemma for manifolds*: Let X be an n-dimensional manifold and $\omega \in Z^k(X)$, $k > 0$ a closed k-form. Then for every point, $p \in X$, there exists a neighborhood, U of p and a $(k-1)$-form $\mu \in \Omega^{k-1}(U)$ such that $\omega = d\mu$ on U. Proof: For open subsets of $\mathbb{R}^n$ we proved this result in §2.3 and since X is locally diffeomorphic at p to an open subset of $\mathbb{R}^n$ this result is true for manifolds as well.

8. Let X be the unit sphere, S^n, in $\mathbb{R}^{n+1}$. Since S^n is compact, connected and oriented

$$H^0(S^n) = H^n(S^n) = \mathbb{R}\,. \tag{5.1.12}$$

We will show that for $k \neq, 0, n$

$$H^k(S^n) = \{0\}\,. \tag{5.1.13}$$

To see this let $\omega \in \Omega^k(S^n)$ be a closed k-form and let $p = (0, \dots, 0, 1) \in S^n$ be the "north pole" of S^n. By the Poincaré lemma there exists a neighborhood, U, of p in S^n and a $k-1$-form, $\mu \in \Omega^{k-1}(U)$ with $\omega = d\mu$ on U. Let $\rho \in \mathcal{C}_0^\infty(U)$ be a "bump function" which is equal to one on a neighborhood, U_0 of U in p. Then

$$\omega_1 = \omega - d\rho\mu \tag{5.1.14}$$

is a closed k-form with compact support in $S^n - \{p\}$. However stereographic projection gives one a diffeomorphism

$$\varphi : \mathbb{R}^n \to S^n - \{p\}$$

(see exercise 1 below), and hence $\varphi^*\omega_1$ is a closed compactly supported k-form on $\mathbb{R}^n$ with support in a large rectangle. Thus by (5.1.14) $\varphi^*\omega = d\nu$, for some $\nu \in \Omega_c^{k-1}(\mathbb{R}^n)$, and by (5.1.14)

$$\omega = d(\rho\mu + (\varphi^{-1})^*\nu) \tag{5.1.15}$$

with $(\varphi^{-1})^*\nu \in \Omega_c^{k-1}(S^n - \{p\}) \subseteq \Omega^k(S^n)$, so we've proved that for $0 < k < n$ every closed k-form on S^n is exact.

We will next discuss some "pull-back" operations in DeRham theory. Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. For $\omega \in \Omega^k(Y)$, $df^*\omega = f^*\,d\omega$, so if ω is closed, $f^*\omega$ is as well. Moreover, if $\omega = d\mu$, $f^*\omega = df^*\mu$, so if ω is exact, $f^*\omega$ is as well. Thus we have linear maps

$$f^* : Z^k(Y) \to Z^k(X) \tag{5.1.16}$$

and

$$f^* : B^k(Y) \to B^k(X) \tag{5.1.17}$$

and comparing (5.1.16) with the projection

$$\pi : Z^k(X) \to Z^k(X)/B^k(X)$$

we get a linear map

$$Z^k(Y) \to H^k(X)\,. \tag{5.1.18}$$

In view of (5.1.17), $B^k(Y)$ is in the kernel of this map, so by Theorem 1.2.2 one gets an induced linear map

$$f^\sharp : H^k(Y) \to H^k(Y)\,, \tag{5.1.19}$$

such that $f^\sharp \circ \pi$ is the map (5.1.18). In other words, if ω is a closed k-form on Y $f^\sharp$ has the defining property

$$f^\sharp[\omega] = [f^*\omega]\,. \tag{5.1.20}$$

This "pull-back"operation on cohomology satisfies the following chain rule: Let Z be a manifold and $g : Y \to Z$ a $\mathcal{C}^\infty$ map. Then if ω is a closed k-form on Z

$$(g \circ f)^*\omega = f^*g^*\omega$$

by the chain rule for pull-backs of forms, and hence by (5.1.20)

$$(g \circ f)^\sharp[\omega] = f^\sharp(g^\sharp[\omega])\,. \tag{5.1.21}$$

The discussion above carries over verbatim to the setting of compactly supported DeRham cohomology: If $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map it induces a pull-back map on cohomology

$$f^\sharp : H^k_c(Y) \to H^k_c(X) \tag{5.1.22}$$

and if $f : X \to Y$ and $g : Y \to Z$ are proper $\mathcal{C}^\infty$ maps then the chain rule (5.1.21) holds for compactly supported DeRham cohomology as well as for ordinary DeRham cohomology. Notice also that if $f : X \to Y$ is a diffeomorphism, we can take Z to be X itself and g to be f^{-1}, and in this case the chain rule tells us that the maps (5.1.19) and (5.1.22) are bijections, i.e., $H^k(X)$ and $H^k(Y)$ and $H^k_c(X)$ and $H^k_c(Y)$ are isomorphic as vector spaces.

We will next establish an important fact about the pull-back operation, $f^\sharp$; we'll show that it's a *homotopy invariant* of f. Recall that two $\mathcal{C}^\infty$ maps

$$f_i : X \to Y\,, \quad i = 0, 1 \tag{5.1.23}$$

are homotopic if there exists a $\mathcal{C}^\infty$ map

$$F : X \times [0,1] \to Y$$

with the property $F(p,0) = f_0(p)$ and $F(p,1) = f_1(p)$ for all $p \in X$. We will prove:

Theorem 5.1.1. *If the maps (5.1.23) are homotopic then, for the maps they induce on cohomology*

$$f_0^\sharp = f_1^\sharp\,. \tag{5.1.24}$$

Our proof of this will consist of proving this for an important special class of homotopies, and then by "pull-back" tricks deducing this result for homotopies in general. Let v be a complete vector field on X and let

$$f_t : X \to X\,, \quad -\infty < t < \infty$$

be the one-parameter group of diffeomorphisms it generates. Then

$$F : X \times [0,1] \to X\,, \qquad F(p,t) = f_t(p)\,,$$

is a homotopy between f_0 and f_1, and we'll show that for this homotopic pair (5.1.24) is true. Recall that for $\omega \in \Omega^k(X)$

$$\left(\frac{d}{dt} f_t^* \omega\right)(t=0) = L_v = \iota(v)\, d\omega + \, d\iota(v)\omega$$

and more generally for all t

$$\begin{aligned}
\frac{d}{dt} f_t^* \omega &= \left(\frac{d}{ds} f_{s+t}^* \omega\right)(s=0) \\
&= \left(\frac{d}{ds} (f_s \circ f_t)^* \omega\right)(s=0) \\
&= \left(\frac{d}{ds} f_t^* f_s^* \omega\right)(s=0) = f_t^* \left(\frac{d}{ds} f_s^* \omega\right)(s=0) \\
&= f_t^* L_v \omega \\
&= f_t^* \iota(v)\, d\omega + \, d f_t^* \iota(v) \omega \,.
\end{aligned}$$

Thus if we set

$$Q_t \omega = f_t^* \iota(v) \omega \tag{5.1.25}$$

we get from this computation:

$$\frac{d}{dt} f^* \omega = \, dQ_t + Q_t \, d\omega \tag{5.1.26}$$

and integrating over $0 \le t \le 1$:

$$f_1^* \omega - f_0^* \omega = \, dQ\omega + Q\, d\omega \tag{5.1.27}$$

where

$$Q : \Omega^k(Y) \to \Omega^{k-1}(X)$$

is the operator

$$Q\omega = \int_0^1 Q_t \omega \, dt \,. \tag{5.1.28}$$

The identity (5.1.24) is an easy consequence of this "chain homotopy" identity. If ω is in $Z^k(X)$, $d\omega = 0$, so

$$f_1^* \omega - f_0^* \omega = \, dQ\omega$$

and

$$f_1^\sharp[\omega] - f_0^\sharp[\omega] = [f_1^*\omega - f_0^*\omega] = 0\,.$$

Q.E.D.

We'll now describe how to extract from this result a proof of Theorem 5.1.1 for *any* pair of homotopic maps. We'll begin with the following useful observation.

Proposition 5.1.2. *If $f_i : X \to Y$, $i = 0, 1$, are homotopic $\mathcal{C}^\infty$ mappings there exists a $\mathcal{C}^\infty$ map*

$$F : X \times \mathbb{R} \to Y$$

such that the restriction of F to $X \times [0, 1]$ is a homotopy between f_0 and f_1.

Proof. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$, $\rho \geq 0$, be a bump function which is supported on the interval, $\frac{1}{4} \leq t \leq \frac{3}{4}$ and is positive at $t = \frac{1}{2}$. Then

$$\chi(t) = \int_{-\infty}^{t} \rho(s)\, ds \Big/ \int_{-\infty}^{\infty} \rho(s)\, ds$$

is a function which is zero on the interval $t \leq \frac{1}{4}$, is one on the interval $t \geq \frac{3}{4}$, and, for all t, lies between 0 and 1. Now let

$$G : X \times [0, 1] \to Y$$

be a homotopy between f_0 and f_1 and let $F : X \times \mathbb{R} \to Y$ be the map

$$F(x, t) = G(x, \chi(t))\,. \tag{5.1.29}$$

This is a $\mathcal{C}^\infty$ map and since

$$F(x, 1) = G(x, \chi(1)) = G(x, 1) = f_1(x)$$

and

$$F(x, 0) = G(x, \chi(0)) = G(x, 0) = f_0(x)\,,$$

it gives one a homotopy between f_0 and f_1.

□

We're now in position to deduce Theorem 5.1.1 from the version of this result that we proved above.

Let

$$\gamma_t : X \times \mathbb{R} \to X \times \mathbb{R}\,, \quad -\infty < t < \infty$$

be the one-parameter group of diffeomorphisms

$$\gamma_t(x,a) = (x, a+t)$$

and let $v = \partial/\partial t$ be the vector field generating this group. For k-forms, $\mu \in \Omega^k(X \times \mathbb{R})$, we have by (5.1.27) the identity

$$\gamma_1^*\mu - \gamma_0^*\mu = d\Gamma\mu + \Gamma\, d\mu \tag{5.1.30}$$

where

$$\Gamma\mu = \int_0^1 \gamma_t^* \left(\iota\left(\frac{\partial}{\partial t}\right)\mu\right) dt\,. \tag{5.1.31}$$

Now let F, as in Proposition 5.1.2, be a $\mathcal{C}^\infty$ map

$$F : X \times \mathbb{R} \to Y$$

whose restriction to $X \times [0,1]$ is a homotopy between f_0 and f_1. Then for $\omega \in \Omega^k(Y)$

$$\gamma_1^* F^*\omega - \gamma_0^* F^*\omega = d\Gamma F^*\mu + \Gamma F^*\, d\mu \tag{5.1.32}$$

by the identity (5.1.29). Now let $\iota : X \to X \times \mathbb{R}$ be the inclusion, $p \to (p,0)$, and note that

$$(F \circ \gamma_1 \circ \iota)(p) = F(p,1) = f_1(p)$$

and

$$(F \circ \gamma_0 \circ \iota)(p) = F(p,0) = f_0(p)$$

i.e.,

$$F \circ \gamma_1 \circ \iota \quad = f_1 \tag{5.1.33}$$

and

$$F \circ \gamma_0 \circ \iota \quad = f_0\,. \tag{5.1.34}$$

Thus

$$\iota^*(\gamma_1^* F^*\omega - \gamma_0^* F^*\omega) = f_1^*\omega - f_0^*\omega$$

and on the other hand by (5.1.31)

$$\iota^*(\gamma_1^* F^*\omega - \gamma_0^* F^*\omega) = d\iota^*\Gamma F^*\omega + \iota^*\Gamma F^*\, d\omega\,.$$

Letting

$$Q : \Omega^k(Y) \to \Omega^{k-1}(X) \tag{5.1.35}$$

be the "chain homotopy" operator

$$Q\omega = \iota^*\Gamma F^*\omega \tag{5.1.36}$$

we can write the identity above more succinctly in the form

$$f_1^*\omega - f_c^*\omega = dQ\omega + Q\, d\omega \tag{5.1.37}$$

and from this deduce, exactly as we did earlier, the identity (5.1.24).

This proof can easily be adapted to the compactly supported setting. Namely the operator (5.1.36) is defined by the integral

$$Q\omega = \int_0^1 \iota^*\gamma_t^* \left(\iota\left(\frac{\partial}{\partial t}\right) F^*\omega\right) dt\,. \tag{5.1.38}$$

Hence if ω is supported on a set, A, in Y, the integrand of (5.1.37) at t is supported on the set

$$\{p \in X\,, \quad F(p,t) \in A\} \tag{5.1.39}$$

and hence $Q\omega$ is supported on the set

$$\pi(F^{-1}(A) \cap X \times [0,1]) \tag{5.1.40}$$

where $\pi : X \times [0,1] \to X$ is the projection map, $\pi(p,t) = p$. Suppose now that f_0 and f_1 are proper mappings and

$$G : X \times [0,1] \to Y$$

a *proper* homotopy between f_0 and f_1, i.e., a homotopy between f_0 and f_1 which is proper as a $\mathcal{C}^\infty$ map. Then if F is the map (5.1.30) its restriction to $X \times [0,1]$ is also a proper map, so this restriction is also a proper homotopy between f_0 and f_1. Hence if ω is in $\Omega_c^k(Y)$ and A is its support, the set (5.1.39) is compact, so $Q\omega$ is in $\Omega_c^{k-1}(X)$. Therefore all summands in the "chain homotopy" formula (5.1.37) are compactly supported. Thus we've proved

Theorem 5.1.3. *If $f_i : X \to Y$, $i = 0,1$ are proper $\mathcal{C}^\infty$ maps which are homotopic via a proper homotopy, the induced maps on cohomology*

$$f_i^\sharp : H_c^k(Y) \to H_c^k(X)$$

are the same.

We'll conclude this section by noting that the cohomology groups, $H^k(X)$, are equipped with a natural product operation. Namely, suppose $\omega_i \in \Omega^{k_i}(X)$, $i = 1,2$, is a closed form and that $c_i = [\omega_i]$ is the cohomology class represented by ω_i. We can then define a product cohomology class $c_1 \cdot c_2$ in $H^{k_1+k_2}(X)$ by the recipe

$$c_1 \cdot c_2 = [\omega_1 \wedge \omega_2] . \tag{5.1.41}$$

To show that this is a legitimate definition we first note that since ω_2 is closed

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge\, d\omega_2 = 0 ,$$

so $\omega_1 \wedge \omega_2$ is closed and hence does represent a cohomology class. Moreover if we replace ω_1 by another representative, $\omega_1 + d\mu_1 = \omega'$, of the cohomology class, c_1

$$\omega_1' \wedge \omega_2 = \omega_1 \wedge \omega_2 + \, d\mu_1 \wedge \omega_2 .$$

But since ω_2 is closed,

$$\begin{aligned} d\mu_1 \wedge \omega_2 &= d(\mu_1 \wedge \omega_2) + (-1)^{k_1} \mu_1 \wedge\, d\omega_2 \\ &= d(\mu_1 \wedge \omega_2) \end{aligned}$$

so

$$\omega_1' \wedge \omega_2 = \omega_1 \wedge \omega_2 + d(\mu_1 \wedge \omega_2)$$

and $[\omega_1' \wedge \omega_2] = [\omega_1 \wedge \omega_2]$. Similary (5.1.41) is unchanged if we replace ω_2 by $\omega_2 + d\mu_2$, so the definition of (5.1.41) depends neither on the choice of ω_1 nor ω_2 and hence is an intrinsic definition as claimed.

There is a variant of this product operation for compactly supported cohomology classes, and we'll leave for you to check that it's also well defined. Suppose c_1 is in $H_c^{k_1}(X)$ and c_2 is in $H^{k_2}(X)$ (i.e., c_1 is a compactly supported class and c_2 is an ordinary cohomology class). Let ω_1 be a representative of c_1 in $\Omega_c^{k_1}(X)$ and ω_2

a representative of c_2 in $\Omega^{k_2}(X)$. Then $\omega_1 \wedge \omega_2$ is a closed form in $\Omega_c^{k_1+k_2}(X)$ and hence defines a cohomology class

$$c_1 \cdot c_2 = [\omega_1 \wedge \omega_2] \tag{5.1.42}$$

in $H_c^{k_1+k_2}(X)$. We'll leave for you to check that this is intrinsically defined. We'll also leave for you to check that (5.1.42) is intrinsically defined if the roles of c_1 and c_2 are reversed, i.e., if c_1 is in $H^{k_1}(X)$ and c_2 in $H_c^{k_2}(X)$ and that the products (5.1.41) and (5.1.42) both satisfy

$$c_1 \cdot c_2 = (-1)^{k_1 k_2} c_2 \cdot c_1 \,. \tag{5.1.43}$$

Finally we note that if Y is another manifold and $f : X \to Y$ a $\mathcal{C}^\infty$ map then for $\omega_1 \in \Omega^{k_1}(Y)$ and $\omega_2 \in \Omega^{k_2}(Y)$

$$f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$$

by (2.5.7) and hence if ω_1 and ω_2 are closed and $c_i = [\omega_i]$

$$f^\sharp(c_1 \cdot c_2) = f^\sharp c_1 \cdot f^\sharp c_2 \,. \tag{5.1.44}$$

Exercises.

1. (Stereographic projection.) Let $p \in S^n$ be the point, $(0, 0, \dots, 0, 1)$. Show that for every point $x = (x_1, \dots, x_{n+1})$ of $S^n - \{p\}$ the ray

$$tx + (1-t)p \,, \quad t > 0$$

intersects the plane, $x_{n+1} = 0$, in the point

$$\gamma(x) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n)$$

and that the map

$$\gamma : S^n - \{p\} \to \mathbb{R}^n \,, \quad x \to \gamma(x)$$

is a diffeomorphism.

2. Show that the operator

$$Q_t : \Omega^k(Y) \to \Omega^{k-1}(X)$$

in the integrand of (5.1.38), i.e., the operator,

$$Q_t\omega = \iota^*\gamma_t^*\left(\iota\left(\frac{\partial}{\partial t}\right)F^*\omega\right)$$

has the following description. Let p be a point of X and let $q = f_t(p)$. The curve, $s \to f_s(p)$ passes through q at time $s = t$. Let $v(q) \in T_qY$ be the tangent vector to this curve at t. Show that

$$(Q_t\omega)(p) = (df_t^*)_p\iota(v_q)\omega_q\,. \tag{5.1.45}$$

3. Let U be a star-shaped open subset of $\mathbb{R}^n$, i.e., a subset of $\mathbb{R}^n$ with the property that for every $p \in U$ the ray, tp, $0 \leq t < 1$, is in U.

(a) Let v be the vector field

$$v = \sum x_i \frac{\partial}{\partial x_i}$$

and $\gamma_t : U \to U$, the map $p \to tp$. Show that for every k-form, $\omega \in \Omega^k(U)$

$$\omega = dQ\omega + Q\,d\omega$$

where

$$Q\omega = \int_0^1 \gamma_t^*\iota(v)\omega\frac{dt}{t}\,.$$

(b) Show that if

$$\omega = \sum a_I(x)\,dx_I$$

then

$$Q\omega = \sum_{I,r}\left(\int t^{k-1}(-1)^{r-1}x_{i_r}a_I(tx)\,dt\right)dx_{I_r} \tag{5.1.46}$$

where

$$dx_{I_r} = dx_{i_1}\wedge\cdots\widehat{dx_{i_r}}\wedge\cdots dx_{i_k}\,.$$

4. Let X and Y be oriented connected n-dimensional manifolds, and $f : X \to Y$ a proper map. Show that the linear map, L, in the diagram below

$$\begin{array}{ccc} H_c^n(Y) & \xrightarrow{f^\sharp} & H_c^n(X) \\ {\scriptstyle I_Y}\downarrow & & {\scriptstyle I_X}\downarrow \\ \mathbb{R} & \xrightarrow{L} & \mathbb{R} \end{array}$$

is just the map, $t \in \mathbb{R} \to \deg(f)t$.

5. Let X and Y be manifolds and let id_X and id_Y be the identity maps of X onto X and Y onto Y. A homotopy equivalence between X and Y is a pair of maps

$$f : X \to Y$$

and

$$g : Y \to X$$

such that $g \circ f$ is homotopic to $i\,d_X$ and $f \circ g$ is homotopic to id_Y. Show that if X and Y are homotopy equivalent their cohomology groups are the same "up to isomorphism", i.e., there exist bijections

$$H^k(X) \to H^k(Y)\,.$$

6. Show that $\mathbb{R}^n - \{0\}$ and S^{n-1} are homotopy equivalent.

7. What are the cohomology groups of the n-sphere with two points deleted? *Hint:* The n-sphere with one point deleted is $\mathbb{R}^n$.

8. Let X and Y be manifolds and $f_i : X \to Y$, $i = 0, 1, 2$, $\mathcal{C}^\infty$ maps. Show that if f_0 and f_1 are homotopic and f_1 and f_2 are homotopic then f_0 and f_2 are homotopic.

Hint: The homotopy (5.1.20) has the property that

$$F(p,t) = f_t(p) = f_0(p)$$

for $0 \leq t \leq \frac{1}{4}$ and

$$F(p,t) = f_t(p) = f_1(p)$$

for $\frac{3}{4} \leq t < 1$. Show that two homotopies with these properties: a homotopy between f_0 and f_1 and a homotopy between f_1 and f_2, are easy to "glue together" to get a homotopy between f_0 and f_2.

9. (a) Let X be an n-dimensional manifold. Given points $p_i \in X$, $i = 0, 1, 2$ show that if p_0 can be joined to p_1 by a $\mathcal{C}^\infty$ curve, $\gamma_0 : [0,1] \to X$, and p_1 can be joined to p_2 by a $\mathcal{C}^\infty$ curve, $\gamma_1 : [0,1] \to X$, then p_0 can be joined to p_2 by a $\mathcal{C}^\infty$ curve, $\gamma : [0,1] \to X$.

Hint: A $\mathcal{C}^\infty$ curve, $\gamma : [0,1] \to X$, joining p_0 to p_2 can be thought of as a homotopy between the maps

$$\gamma_{p_0} : pt \to X\,, \quad pt \to p_0$$

and

$$\gamma_{p_1} : pt \to X\,, \quad pt \to p_1$$

where "*pt*" is the zero-dimensional manifold consisting of a single point.

(b) Show that if a manifold, X, is connected it is arc-wise connected: any two points can by joined by a $\mathcal{C}^\infty$ curve.

10. Let X be a connected n-dimensional manifold and $\omega \in \Omega^1(X)$ a closed one-form.

(a) Show that if $\gamma : [0,1] \to X$ is a $\mathcal{C}^\infty$ curve there exists a partition: $0 = a_0 < a_1 < \cdots < a_n = 1$ of the interval $[0,1]$ and open sets U_i in X such that $\gamma\left([a_{i-1}, a_i]\right) \subseteq U_i$ and such that $\omega | U_i$ is exact.

(b) In part (a) show that there exist functions, $f_i \in \mathcal{C}^\infty(U_i)$ such that $\omega | U_i = df_i$ and $f_i\left(\gamma(a_i)\right) = f_{i+1}\left(\gamma(a_i)\right)$.

(c) Show that if p_0 and p_1 are the end points of γ

$$f_n(p_1) - f_1(p_0) = \int_0^1 \gamma^*\omega\,.$$

(d) Let

$$\gamma_s : [0,1] \to X\,, \quad 0 \le s \le 1 \tag{5.1.47}$$

be a homotopic family of curves with $\gamma_s(0) = p_0$ and $\gamma_s(1) = p_1$. Prove that the integral

$$\int_0^1 \gamma_s^*\omega$$

is independent of s_0. *Hint:* Let s_0 be a point on the interval, $[0,1]$. For $\gamma = \gamma_{s_0}$ choose a_i's and f_i's as in parts (a)–(b) and show that for s close to s_0, $\gamma_s[a_{i-1}, a_i] \subseteq U_i$.

(e) A manifold, X, is simply connected if, for any two curves, $\gamma_i : [0,1] \to X$, $i = 0, 1$, with the same end-points, p_0 and p_1, there exists a homotopy (5.1.42) with $\gamma_s(0) = p_0$ and $\gamma_s(1) = p_1$, i.e., γ_0 can be smoothly deformed into γ_1 by a family of curves all having the same end-points. Prove

Theorem 5.1.4. *If X is simply-connected $H^1(X) = \{0\}$.*

11. Show that the product operation (5.1.41) is associative and satisfies left and right distributive laws.

12. Let X be a compact oriented $2n$-dimensional manifold. Show that the map

$$B : H^n(X) \times H^n(X) \to \mathbb{R}$$

defined by

$$B(c_1, c_2) = I_X(c_1 \cdot c_2)$$

is a bilinear form on $H^n(X)$ and that it's symmetric if n is even and alternating if n is odd.

5.2 The Mayer–Victoris theorem

In this section we'll develop some techniques for computing cohomology groups of manifolds. (These techniques are known collectively as "diagram chasing" and the mastering of these techniques is more akin to becoming proficient in checkers or chess or the Sunday acrostics in the *New York Times* than in the areas of mathematics to which they're applied.) Let C^i, $i = 0, 1, 2, \ldots$, be vector spaces and $d : C^i \to C^{i+1}$ a linear map. The sequence of vector spaces and maps

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots \tag{5.2.1}$$

is called a *complex* if $d^2 = 0$, i.e., if for $a \in C^k$, $d(da) = 0$. For instance if X is a manifold the DeRham complex

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots \tag{5.2.2}$$

is an example of a complex, and the complex of compactly supported DeRham forms

$$\Omega^0_c(X) \xrightarrow{d} \Omega^1_c(X) \xrightarrow{d} \Omega^2_c(X) \to \cdots \tag{5.2.3}$$

is another example. One defines the *cohomology groups* of the complex (5.2.1) in exactly the same way that we defined the cohomology groups of the complexes (5.2.2) and (5.2.3) in §5.1. Let

$$Z^k = \{a \in C^k \,;\, da = 0\}$$

and

$$B^k = \{a \in C^k \,;\, a \in dC^{k-1}\}$$

i.e., let a be in B^k if and only if $a = db$ for some $b \in C^{k-1}$. Then $da = d^2 b = 0$, so B^k is a vector subspace of Z^k, and we define $H^k(C)$ — the k^{th} cohomology group of the complex (5.2.1) — to be the quotient space

$$H^k(C) = Z^k / B^k \,. \tag{5.2.4}$$

Given $c \in Z^k$ we will, as in §5.1, denote its image in $H^k(C)$ by $[c]$ and we'll call c a *representative* of the cohomology class $[c]$.

We will next assemble a small dictionary of "diagram chasing" terms.

Definition 5.2.1. *Let V_i, $i = 0, 1, 2, \ldots$, be vector spaces and $\alpha_i : V_i \to V_{i+1}$ linear maps. The sequence*

$$V_0 \stackrel{\alpha_0}{\to} V_1 \stackrel{\alpha_1}{\to} V_2 \stackrel{\alpha_2}{\to} \cdots \tag{5.2.5}$$

is an exact sequence *if, for each i, the kernel of α_{i+1} is equal to the image of α_i.*

For example the sequence (5.2.1) is exact if $Z_i = B_i$ for all i, or, in other words, if $H^i(C) = 0$ for all i. A simple example of an exact sequence that we'll encounter a lot below is a sequence of the form

$$\{0\} \to V_1 \stackrel{\alpha_1}{\to} V_2 \stackrel{\alpha_2}{\to} V_3 \to \{0\} \,, \tag{5.2.6}$$

a five term exact sequence whose first and last terms are the vector space, $V_0 = V_4 = \{0\}$, and hence $\alpha_0 = \alpha_3 = 0$. This sequence is exact if and only if

1. α_1 is injective,
2. the kernel of α_2 equals the image of α_1, and

3. α_2 is surjective.

We will call an exact sequence of this form a *short exact* sequence. (We'll also encounter a lot below an even shorter example of an exact sequence, namely a sequence of the form

$$\{0\} \to V_1 \overset{\alpha_1}{\to} V_2 \to \{0\}\,. \tag{5.2.7}$$

This is an exact sequence if and only if α_1 is bijective.)

Another basic notion in the theory of diagram chasing is the notion of a commutative diagram. The square diagram of vector spaces and linear maps

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ C & \xrightarrow{g} & D \end{array}$$

is commutative if $f \circ i = j \circ g$, and a more complicated diagram of vector spaces and linear maps like the diagram below

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ \uparrow & & \uparrow & & \uparrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \\ \uparrow & & \uparrow & & \uparrow \\ C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 \end{array}$$

is commutative if every subsquare in the diagram, for instance the square,

$$\begin{array}{ccc} B_2 & \longrightarrow & B_3 \\ \uparrow & & \uparrow \\ C_2 & \longrightarrow & C_3 \end{array}$$

is commutative.

We now have enough "diagram chasing" vocabulary to formulate the Mayer–Victoris theorem. For $r = 1, 2, 3$ let

$$\{0\} \to C_r^0 \overset{d}{\to} C_r^1 \overset{d}{\to} C_r^2 \overset{d}{\to} \cdots \tag{5.2.8}$$

be a complex and, for fixed k, let

$$\{0\} \to C_1^k \overset{i}{\to} C_2^k \overset{j}{\to} C_3^k \to \{0\} \tag{5.2.9}$$

be a short exact sequence. Assume that the diagram below commutes:

(5.2.10)
$$\begin{array}{ccccccccc} & & \big| & & \big| & & \big| & & \\ 0 & \longrightarrow & C_1^{k+1} & \xrightarrow{i} & C_2^{k+1} & \xrightarrow{j} & C_3^{k+1} & \longrightarrow & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \longrightarrow & C_1^{k} & \longrightarrow & C_2^{k} & \longrightarrow & C_3^{k} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C_1^{k-1} & \xrightarrow{i} & C_2^{k-1} & \xrightarrow{j} & C_3^{k-1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \end{array}$$

i.e., assume that in the left hand squares, $di = id$, and in the right hand squares, $dj = jd$.

The Mayer–Victoris theorem addresses the following question: If one has information about the cohomology groups of two of the three complexes, (5.2.8), what information about the cohomology groups of the third can be extracted from this diagram? Let's first observe that the maps, i and j, give rise to mappings between these cohomology groups. Namely, for $r = 1, 2, 3$ let Z_r^k be the kernel of the map, $d : C_r^k \to C_r^{k+1}$, and B_r^k the image of the map, $d : C_r^{k-1} \to C_r^k$. Since $id = di$, i maps B_1^k into B_2^k and Z_1^k into Z_2^k, therefore by (5.2.4) it gives rise to a linear mapping

$$i_\sharp : H^k(C_1) \to H^k(C_2)\,.$$

Similarly since $jd = dj$, j maps B_2^k into B_3^k and Z_2^k into Z_3^k, and so by (5.2.4) gives rise to a linear mapping

$$j_\sharp : H^k(C_2) \to H^k(C_3)\,.$$

Moreover, since $j \circ i = 0$ the image of $i_\sharp$ is contained in the kernel of $j_\sharp$. We'll leave as an exercise the following sharpened version of this observation:

Proposition 5.2.2. *The kernel of $j_\sharp$ equals the image of $i_\sharp$, i.e., the three term sequence*

(5.2.11)
$$H^k(C_1) \xrightarrow{i_\sharp} H^k(C_2) \xrightarrow{j_\sharp} H^k(C_3)$$

is exact.

Since (5.2.9) is a short exact sequence one is tempted to conjecture that (5.2.11) is also a short exact sequence (which, if it were true, would tell us that the cohomology groups of any two of the complexes (5.2.8) completely determine the cohomology groups of the third). Unfortunately, this is not the case. To see how this conjecture can be violated let's try to show that the mapping $j_\sharp$ is surjective. Let c_3^k be an element of Z_3^k representing the cohomology class, $[c_3^k]$, in $H^3(C_3)$. Since (5.2.9) is exact there exists a c_2^k in C_2^k which gets mapped by j onto c_3^k, and if c_3^k were in Z_2^k this would imply

$$j_\sharp[c_2^k] = [jc_2^k] = [c_3^k]\,,$$

i.e., the cohomology class, $[c_3^k]$, would be in the image of $j_\sharp$. However, since there's no reason for c_2^k to be in Z_2^k, there's also no reason for $[c_3^k]$ to be in the image of $j_\sharp$. What we can say, however, is that $j\,dc_2^k = djc_2^k = dc_3^k = 0$ since c_3^k is in Z_3^k. Therefore by the exactness of (5.2.9) in degree $k+1$ there exists a unique element, c_1^{k+1} in C_1^{k+1} with property

$$dc_2^k = ic_1^{k+1}\,. \tag{5.2.12}$$

Moreover, since $0 = d(dc_2^k) = di(c_1^{k+1}) = i\,dc_1^{k+1}$ and i is injective, $dc_1^{k+1} = 0$, i.e.,

$$c_1^{k+1} \in Z_1^{k+1}\,. \tag{5.2.13}$$

Thus via (5.2.12) and (5.2.13) we've converted an element, c_3^k, of Z_3^k into an element, c_1^{k+1}, of Z_1^{k+1} and hence set up a correspondence

$$c_3^k \in Z_3^k \to c_1^{k+1} \in Z_1^{k+1}\,. \tag{5.2.14}$$

Unfortunately this correspondence isn't, strictly speaking, a map of Z_3^k into Z_1^{k+1}; the c_1^k in (5.2.14) isn't determined by c_3^k alone but also by the choice we made of c_2^k. Suppose, however, that we make another choice of a c_2^k with the property $j(c_2^k) = c_3^k$. Then the difference between our two choices is in the kernel of j and hence, by the exactness of (2.5.8) at level k, is in the image of i. In other words, our two choices are related by

$$(c_2^k)_{\mathrm{new}} = (c_2^k)_{old} + i(c_1^k)$$

for some c_1^k in C_1^k, and hence by (5.2.12)

$$(c_1^{k+1})_{\mathrm{new}} = (c_1^{k+1})_{old} + dc_1^k\,.$$

Therefore, even though the correspondence (5.2.14) isn't strictly speaking a map it does give rise to a well-defined map

$$Z_3^k \to H^{k+1}(C_1)\,, \quad c_3^k \to [c_3^{k+1}]\,. \tag{5.2.15}$$

Moreover, if c_3^k is in B_3^k, i.e., $c_3^k = dc_3^{k-1}$ for some $c_3^{k-1} \in C_3^{k-1}$, then by the exactness of (5.2.8) at level $k-1$, $c_3^{k-1} = j(c_2^{k-1})$ for some $c_2^{k-1} \in C_2^{k-1}$ and hence $c_3^k = j(dc_2^{k-2})$. In other words we can take the c_2^k above to be dc_2^{k-1} in which case the c_1^{k+1} in equation (5.2.12) is just zero. Thus the map (5.2.14) maps B_3^k to zero and hence by Proposition 1.2.2 gives rise to a well-defined map

$$\delta : H^k(C_3) \to H^{k+1}(C_1) \tag{5.2.16}$$

mapping $[c_3^k] \to [c_1^{k+1}]$. We will leave it as an exercise to show that this mapping measures the failure of the arrow $j_\sharp$ in the exact sequence (5.2.11) to be surjective (and hence the failure of this sequence to be a short exact sequence at its right end).

Proposition 5.2.3. *The image of the map* $j_\sharp : H^k(C_2) \to H^k(C_3)$ *is equal to the kernel of the map,* $\delta : H^k(C_3) \to H^{k+1}(C_1)$.

Hint: Suppose that in the correspondence (5.2.14) c_1^{k+1} is in B_1^{k+1}. Then $c_1^{k+1} = dc_1^k$ for some c_1^k in C_1^k. Show that

$$j(c_2^k - i(c_1^k)) = c_3^k$$

and

$$d(c_2^k - i(c_1^k)) = 0$$

i.e., $c_2^k - i(c_1^k)$ is in Z_2^k and hence $j_\sharp[c_2^k - i(c_1^k)] = [c_3^k]$.

Let's next explore the failure of the map, $i_\sharp : H^{k+1}(C_1) \to H^{k+1}(C_2)$, to be injective. Let c_1^{k+1} be in Z_1^{k+1} and suppose that its cohomology class, $[c_1^{k+1}]$, gets mapped by $i_\sharp$ into zero. This translates into the statement

$$i(c_1^{k+1}) = dc_2^k \tag{5.2.17}$$

for some $c_2^k \in C_2^k$. Moreover since $dc_2^k = i(c_1^{k+1})$, $j(dc_2^k) = 0$. But if

$$c_3^k \stackrel{\text{def}}{=} j(c_2^k) \tag{5.2.18}$$

then $dc_3^k = dj(c_2^k) = j(dc_2^k) = j(i(c_1^{k+1})) = 0$, so c_3^k is in Z_3^k, and by (5.2.17), (5.2.18) and the definition of δ

$$[c_1^{k+1}] = \delta[c_3^k]\,. \tag{5.2.19}$$

In other words the kernel of the map, $i_\sharp : H^{k+1}(C_1) \to H^{k+1}(C_2)$ is contained in the image of the map $\delta : H^k(C_3) \to H^{k+1}(C_1)$. We will leave it as an exercise to show that this argument can be reversed to prove the converse assertion and hence to prove

Proposition 5.2.4. *The image of the map $\delta : H^k(C_1) \to H^{k+1}(C_1)$ is equal to the kernel of the map $i_\sharp : H^{k+1}(C_1) \to H^{k+1}(C_2)$.*

Putting together the Propositions 5.2.2–5.2.4 we obtain the main result of this section: the Mayer–Victoris theorem. The sequence of cohomology groups and linear maps

$$\cdots \xrightarrow{\delta} H^k(C_1) \xrightarrow{i_\sharp} H^k(C_2) \xrightarrow{j_\sharp} H^k(C_3) \xrightarrow{\delta} H^{k+1}(C{-}1) \xrightarrow{i_\sharp} \cdots \tag{5.2.20}$$

is exact.

Remark 5.2.5. *In view of the "$\cdots$"'s this sequence can be a* very *long sequence and is commonly referred to as the "long exact sequence in cohomology" associated to the short exact sequence of complexes (2.5.9).*

Before we discuss the applications of this result, we will introduce some vector space notation. Given vector spaces, V_1 and V_2 we'll denote by $V_1 \oplus V_2$ the vector space sum of V_1 and V_2, i.e., the set of all pairs

$$(u_1, u_2)\,, \quad u_i \in V_i$$

with the addition operation

$$(u_1, u_2) + (\mathrm{v}_1 + \mathrm{v}_2) = (u_1 + \mathrm{v}_1\,,\ u_2 + \mathrm{v}_2)$$

and the scalar multiplication operation

$$\lambda(u_1, u_2) = (\lambda u_1\,,\ \lambda u_2)\,.$$

Now let X be a manifold and let U_1 and U_2 be open subsets of X. Then one has a linear map

$$\Omega^k(U_1 \cup U_2) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \tag{5.2.21}$$

defined by

$$\omega \to (\omega|U_1\,,\,\omega|U_2) \tag{5.2.22}$$

where $\omega|U_i$ is the restriction of ω to U_i. Similarly one has a linear map

$$\Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \tag{5.2.23}$$

defined by

$$(\omega_1, \omega_2) \to \omega_1|U_1 \cap U_2 - \omega_2|U_1 \cap U_2\,. \tag{5.2.24}$$

We claim

Theorem 5.2.6. *The sequence*

$$\{0\} \to \Omega^k(U_1 \cup U_2) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \to \{0\} \tag{5.2.25}$$

is a short exact sequence.

Proof. If the right hand side of (5.2.22) is zero, ω itself has to be zero so the map (5.2.22) is injective. Moreover, if the right hand side of (5.2.24) is zero, ω_1 and ω_2 are equal on the overlap, $U_1 \cap U_2$, so we can glue them together to get a $\mathcal{C}^\infty$ k-form on $U_1 \cup U_2$ by setting $\omega = \omega_1$ on U_1 and $\omega = \omega_2$ on U_2. Thus by (5.2.22) $i(\omega) = (\omega_1, \omega_2)$, and this shows that the kernel of j is equal to the image of i. Hence to complete the proof we only have to show that j is surjective, i.e., that every form ω on $\Omega^k(U_1 \cap U_2)$ can be written as a difference, $\omega_1|U_1 \cap U_2 - \omega_2|U_1 \cap U_2$, where ω_1 is in $\Omega^k(U_1)$ and ω_2 in in $\Omega^k(U_2)$. To prove this we'll need the following variant of the partition of unity theorem.

Theorem 5.2.7. *There exist functions, $\varphi_\alpha \in \mathcal{C}^\infty(U_1 \cup U_2)$, $\alpha = 1, 2$, such that support φ_α is contained in U_α and $\varphi_1 + \varphi_2 = 1$.*

Before proving this let's use it to complete our proof of Theorem 5.2.6. Given $\omega \in \Omega^k(U_1 \cap U_2)$ let

$$\omega_1 = \begin{cases} \varphi_2\omega & \text{on} \quad U_1 \cap U_2 \\ 0 & \text{on} \quad U_1 - U_1 \cap U_2 \end{cases} \tag{5.2.26}$$

and let

(5.2.27) $$\omega_2 = \begin{cases} -\varphi_1\omega & \text{on} \quad U_1 \cap U_2 \\ 0 & \text{on} \quad U_2 - U_1 \cap U_2\,. \end{cases}$$

Since φ_2 is supported on U_2 the form defined by (5.2.26) is $\mathcal{C}^\infty$ on U_1 and since φ_1 is supported on U_1 the form defined by (5.2.27) is $\mathcal{C}^\infty$ on U_2 and since $\varphi_1 + \varphi_2 = 1$, $\omega_1 - \omega_2 = (\varphi_1 + \varphi_2)\omega = \omega$ on $U_1 \cap U_2$. □

To prove Theorem 5.2.7, let $\rho_i \in \mathcal{C}_0^\infty(U_1 \cup U_2)$, $i = 1, 2, 3, \ldots$ be a partition of unity subordinate to the cover, $\{U_\alpha\,,\ \alpha = 1, 2\}$ of $U_1 \cup U_2$ and let φ_1 be the sum of the ρ_i's with support on U_1 and φ_2 the sum of the remaining ρ_i's. It's easy to check (using part (b) of Theorem 4.6.1) that φ_α is supported in U_α and (using part (c) of Theorem 4.6.1) that $\varphi_1 + \varphi_2 = 1$. □

Now let

(5.2.28) $$\{0\} \to C_1^0 \xrightarrow{d} C_1^1 \xrightarrow{d} C_1^2 \to \cdots$$

be the DeRham complex of $U_1 \cup U_2$, let

(5.2.29) $$\{0\} \to C_3^0 \xrightarrow{d} C_3^1 \xrightarrow{d} C_3^2 \to \cdots$$

be the DeRham complex of $U_1 \cap U_2$ and let

(5.2.30) $$\{0\} \to C_2^0 \xrightarrow{d} C_2^1 \xrightarrow{d} C_2^2 \xrightarrow{d} \cdots$$

be the vector space direct sum of the DeRham complexes of U_1 and U_2, i.e., the complex whose k^{th} term is

$$C_2^k = \Omega^k(U_1) \oplus \Omega^k(U_2)$$

with $d : C_2^k \to C_2^{k+1}$ defined to be the map $d(\mu_1, \mu_2) = (d\mu_1,\, d\mu_2)$. Since $C_1^k = \Omega^k(U_1 \cup U_2)$ and $C_3^k = \Omega^k(U_1 \cap U_2)$ we have, by Theorem 5.2.6, a short exact sequence

(5.2.31) $$\{0\} \to C_1^k \xrightarrow{i} C_2^k \xrightarrow{j} C_3^k \to \{0\}\,,$$

and it's easy to see that i and j commute with the d's:

(5.2.32) $$di = id \text{ and } dj = jd\,.$$

Hence we're exactly in the situation to which Mayer–Victoris applies. Since the cohomology groups of the complexes (5.2.28) and (5.2.29) are the DeRham cohomology group. $H^k(U_1 \cup U_2)$ and $H^k(U_1 \cap U_2)$, and the cohomology groups of the complex (5.2.30) are the vector space direct sums, $H^k(U_1) \oplus H^k(U_2)$, we obtain from the abstract Mayer–Victoris theorem, the following DeRham theoretic version of Mayer–Victoris.

Theorem 5.2.8. *Letting $U = U_1 \cup U_2$ and $V = U_1 \cap U_2$ one has a long exact sequence in DeRham cohomology:*
(5.2.33)

$$\cdots \xrightarrow{\delta} H^k(U) \xrightarrow{i_\sharp} H^k(U_1) \oplus H^k(U_2) \xrightarrow{j_\sharp} H^k(V) \xrightarrow{\delta} H^{k+1}(U) \xrightarrow{i_\sharp} \cdots .$$

This result also has an analogue for compactly supported DeRham cohomology. Let

(5.2.34) $$i : \Omega_c^k(U_1 \cap U_2) \to H_c^k(U_1) \oplus \Omega_c^k(U_2)$$

be the map

(5.2.35) $$i(\omega) = (\omega_1, \omega_2)$$

where

(5.2.36) $$\omega_i = \begin{cases} \omega & \text{on} \quad U_1 \cap U_2 \\ 0 & \text{on} \quad U_i - U_1 \cap U_2 \,. \end{cases}$$

(Since ω is compactly supported on $U_1 \cap U_2$ the form defined by (5.2.34) is a $\mathcal{C}^\infty$ form and is compactly supported on U_i.) Similarly, let

(5.2.37) $$j : \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \to \Omega_c^k(U_1 \cup U_2)$$

be the map

(5.2.38) $$j(\omega_1, \omega_2) = \widetilde{\omega}_1 - \widetilde{\omega}_2$$

where:

(5.2.39) $$\widetilde{\omega}_i = \begin{cases} \omega_i & \text{on} \quad U_i \\ 0 & \text{on} \quad (U_1 \cup U_2) - U_i \,. \end{cases}$$

As above it's easy to see that i is injective and that the kernel of j is equal to the image of i. Thus if we can prove that j is surjective we'll have proved

Theorem 5.2.9. *The sequence*

(5.2.40)
$$\{0\} \to \Omega_c^k(U_1 \cap U_2) \xrightarrow{i} \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \xrightarrow{j} \Omega_c^k(U_1 \cap U_2) \to \{0\}$$

is a short exact sequence.

Proof. To prove the surjectivity of j we mimic the proof above. Given ω in $\Omega_c^k(U_1 \cup U_2)$ let

(5.2.41)
$$\omega = \varphi_1 \omega | U_1$$

and

(5.2.42)
$$\omega_2 = -\varphi_2 \omega | U_2 \,.$$

Then by (5.2.36) $\omega = j(\omega_1, \omega_2)$.

□

Thus, applying Mayer–Victoris to the compactly supported versions of the complexes (5.2.8), we obtain:

Theorem 5.2.10. *Letting* $U = U_1 \cup U_2$ *and* $V = U_1 \cap U_2$ *there exists a long exact sequence in compactly supported DeRham cohomology*

(5.2.43)
$$\cdots \xrightarrow{\delta} H_c^k(V) \xrightarrow{i_\sharp} H_c^k(U_1) \oplus H_c^k(U_2) \xrightarrow{j_\sharp} H_c^k(U) \xrightarrow{\delta} H_c^{k+1}(V) \xrightarrow{i_\sharp} \cdots \,.$$

Exercises

1. Prove Proposition 5.2.2.

2. Prove Proposition 5.2.3.

3. Prove Proposition 5.2.4.

4. Show that if U_1, U_2 and $U_1 \cap U_2$ are non-empty and connected the first segment of the Mayer–Victoris sequence is a short exact sequence

$$\{0\} \to H^0(U_1 \cup U_2) \to H^0(U_1) \oplus H^0(U_2) \to H^0(U_1 \cap U_2) \to \{0\} \,.$$

5. Let $X = S^n$ and let U_1 and U_2 be the open subsets of S^n obtained by removing from S^n the points, $p_1 = (0, \dots, 0, 1)$ and $p_2 = (0, \dots, 0, -1)$.

(a) Using stereographic projection show that U_1 and U_2 are diffeomorphic to $\mathbb{R}^n$.

(b) Show that $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is homotopy equivalent to S^{n-1}. (See problem 5 in §5.1.) *Hint:* $U_1 \cap U_2$ is diffeomorphic to $\mathbb{R}^n - \{0\}$.

(c) Deduce from the Mayer–Victoris sequence that $H^{i+1}(S^n) = H^i(S^{n-1})$ for $i \geq 1$.

(d) Using part (c) give an inductive proof of a result that we proved by other means in §5.1: $H^k(S^n) = \{0\}$ for $1 \leq k < n$.

6. Using the Mayer–Victoris sequence of exercise 5 with cohomology replaced by compactly supported cohomology show that

$$H_c^k(\mathbb{R}^n - \{0\}) \quad \cong \mathbb{R}$$

for $k = 1$ and n and

$$H_c^k(\mathbb{R}^n - \{0\}) \quad = \{0\}$$

for all other values of k.

5.3 Good covers

In this section we will show that for compact manifolds (and for lots of other manifolds besides) the DeRham cohomology groups which we defined in §5.1 are finite dimensional vector spaces and thus, in principle, "computable" objects. A key ingredient in our proof of this fact is the notion of a *good cover* of a manifold.

Definition 5.3.1. *Let X be an n-dimensional manifold, and let*

$$\mathbb{U} = \{U_\alpha, \alpha \in \mathcal{I}\}$$

be a covering of X by open sets. This cover is a good *cover if for every finite set of indices, $\alpha_i \in \mathcal{I}$, $i = 1, \ldots, k$, the intersection $U\alpha_1 \cap \cdots \cap U\alpha_k$ is either empty or is diffeomorphic to $\mathbb{R}^n$.*

One of our first goals in this section will be to show that good covers exist. We will sketch below a proof of the following.

Theorem 5.3.2. *Every manifold admits a good cover.*

The proof involves an elementary result about open convex subsets of $\mathbb{R}^n$.

Proposition 5.3.3. *If U is a bounded open convex subset of $\mathbb{R}^n$, it is diffeomorphic to $\mathbb{R}^n$.*

A proof of this will be sketched in exercises 1–4 at the end of this section.

One immediate consequence of this result is an important special case of Theorem 5.3.2.

Theorem 5.3.4. *Every open subset, U, of $\mathbb{R}^n$ admits a good cover.*

Proof. For each $p \in U$ let U_p be an open convex neighborhood of p in U (for instance an ϵ-ball centered at p) . Since the intersection of any two convex sets is again convex the cover, $\{U_p, p \in U\}$ is a good cover by Proposition 5.3.3.

□

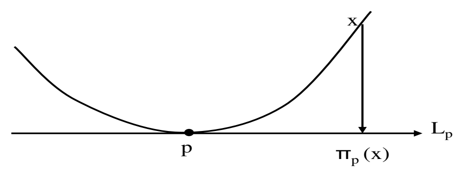
For manifolds the proof of Theorem 5.3.2 is somewhat trickier. The proof requires a manifold analogue of the notion of convexity and there are several serviceable candidates. The one we will use is the following. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold and for $p \in X$ let T_pX be the tangent space to X at p. Recalling that T_pX sits inside $T_p\mathbb{R}^N$ and that

$$T_p\mathbb{R}^N = \{(p, \mathrm{v}) \,,\ \mathrm{v} \in \mathbb{R}^N\}$$

we get a map

$$T_pX \hookrightarrow T_p\mathbb{R}^N \to \mathbb{R}^N \,, \quad (p, x) \to p + x \,,$$

and this map maps T_pX bijectively onto an n-dimensional "affine" subspace, L_p, of $\mathbb{R}^N$ which is tangent to X at p. Let $\pi_p : X \to L_p$ be, as in the figure below, the orthogonal projection of X onto L_p.



Definition 5.3.5. *An open subset, V, of X is* convex *if for every $p \in V$ the map $\pi_p : X \to L_p$ maps V diffeomorphically onto a convex open subset of L_p.*

It's clear from this definition of convexity that the intersection of two open convex subsets of X is an open convex subset of X and that every open convex subset of X is diffeomorphic to $\mathbb{R}^n$. Hence to prove Theorem 5.3.2 it suffices to prove that every point, p, in X is contained in an open convex subset, U_p, of X. Here is a sketch of how to prove this. In the figure above let $B^\epsilon(p)$ be the ball of radius ϵ about p in L_p centered at p. Since L_p and T_p are tangent at p the derivative of π_p at p is just the identity map, so for ϵ small π_p maps a neighborhood, U_p^ϵ of p in X diffeomorphically onto $B^\epsilon(p)$. We claim

Proposition 5.3.6. *For ϵ small, U_p^ϵ is a convex subset of X.*

Intuitively this assertion is pretty obvious: if q is in U_p^ϵ and ϵ is small the map

$$B_p^\epsilon \xrightarrow{\pi_p^{-1}} U_p^\epsilon \xrightarrow{\pi_q} L_q$$

is to order ϵ^2 equal to the identity map, so it's intuitively clear that its image is a slightly warped, but still convex, copy of $B^\epsilon(p)$. We won't, however, bother to write out the details that are required to make this proof rigorous.

A good cover is a particularly good "good cover" if it is a finite cover. We'll codify this property in the definition below.

Definition 5.3.7. *An n-dimensional manifold is said to have* finite topology *if it admits a finite covering by open sets, $U_1, \ldots, U_N$ with the property that for every multi-index, $I = (i_1, \ldots, i_k)$, $1 \le i_1 \le i_2 \cdots < i_K \le N$, the set*

$$U_I = Ui_1 \cap \cdots \cap Ui_k \tag{5.3.1}$$

is either empty or is diffeomorphic to $\mathbb{R}^n$.

If X is a compact manifold and $\mathbb{U} = \{U_\alpha \,,\, \alpha \in \mathcal{I}\}$ is a good cover of X then by the Heine–Borel theorem we can extract from $\mathbb{U}$ a finite subcover

$$U_i = U_{\alpha_i} \,,\, \alpha_i \in \mathcal{I} \,,\, i = 1, \ldots, N \,,$$

hence we conclude

Theorem 5.3.8. *Every compact manifold has finite topology.*

More generally, for any manifold, X, let C be a compact subset of X. Then by Heine–Borel we can extract from the cover, $\mathbb{U}$, a finite subcollection

$$U_i = U_{\alpha_i} \,, \quad \alpha_i \in \mathcal{I} \,, \quad i = 1, \ldots, N$$

that covers C, hence letting $U = \bigcup U_i$, we've proved

Theorem 5.3.9. *If X is an n-dimensional manifold and C a compact subset of X, then there exists an open neighborhood, U, of C in X having finite topology.*

We can in fact even strengthen this further. Let U_0 be any open neighborhood of C in X. Then in the theorem above we can replace X by U_0 to conclude

Theorem 5.3.10. *Let X be a manifold, C a compact subset of X and U_0 an open neighborhood of C in X. Then there exists an open neighborhood, U, of C in X, U contained in U_0, having finite topology.*

We will justify the term "finite topology" by devoting the rest of this section to proving

Theorem 5.3.11. *Let X be an n-dimensional manifold. If X has finite topology the DeRham cohomology groups, $H^k(X)$, $k = 0, \ldots, n$ and the compactly supported DeRham cohomology groups, $H^k_c(X)$, $k = 0, \ldots, n$ are finite dimensional vector spaces.*

The basic ingredients in the proof of this will be the Mayer–Victoris techniques that we developed in §5.2 and the following elementary result about vector spaces.

Lemma 5.3.12. *Let V_i, $i = 1, 2, 3$, be vector spaces and*

$$V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \tag{5.3.2}$$

an exact sequence of linear maps. Then if V_1 and V_3 are finite dimensional, so is V_2.

Proof. Since V_3 is finite dimensional, the image of β is of dimension, $k < \infty$, so there exist vectors, v_i, $i = 1, \ldots, k$ in V_2 having the property that

$$\mathrm{Image}\,\beta = \mathrm{span}\,\{\beta(\mathrm{v}_i)\,, \quad i = 1, \ldots, k\}. \tag{5.3.3}$$

Now let v be any vector in V_2. Then $\beta(\mathrm{v})$ is a linear combination

$$\beta(\mathrm{v}) = \sum_{i=1}^{k} c_i \beta(\mathrm{v}_i) \quad c_i \in \mathbb{R}$$

of the vectors $\beta(\mathrm{v}_i)$ by (5.3.3), so

$$\mathrm{v}' = \mathrm{v} - \sum_{i=1}^{k} c_i \mathrm{v}_i \tag{5.3.4}$$

is in the kernel of β and hence, by the exactness of (5.3.2), in the image of α. But V_1 is finite dimensional, so $\alpha(V_1)$ is finite dimensional. Letting $\mathrm{v}_{k+1}, \ldots, \mathrm{v}_m$ be a basis of $\alpha(V_1)$ we can by (5.3.4) write v as a sum, $\mathrm{v} = \sum_{i=1}^{m} c_i \mathrm{v}_i$. In other words $\mathrm{v}_1, \ldots, \mathrm{v}_m$ is a basis of V_2. □

We'll now prove Theorem 5.3.4. Our proof will be by induction on the number of open sets in a good cover of X. More specifically let

$$\mathbb{U} = \{U_i\,, \, i = 1, \ldots, N\}$$

be a good cover of X. If $N = 1$, $X = U_1$ and hence X is diffeomorphic to $\mathbb{R}^n$, so

$$H^k(X) = \{0\} \text{ for } k > 0$$

and $H^k(X) = \mathbb{R}$ for $k = 0$, so the theorem is certainly true in this case. Let's now prove it's true for arbitrary N by induction. Let U be the open subset of X obtained by forming the union of $U_2, \dots, U_N$. We can think of U as a manifold in its own right, and since $\{U_i\,,\, i = 2, \dots, N\}$ is a good cover of U involving only $N-1$ sets, its cohomology groups are finite dimensional by the induction assumption. The same is also true of the intersection of U with U_1. It has the $N-1$ sets, $U \cap U_i$, $i = 2, \dots, N$ as a good cover, so its cohomology groups are finite dimensional as well. To prove that the theorem is true for X we note that $X = U_1 \cup U$ and that one has an exact sequence

$$H^{k-1}(U_1 \cap U) \xrightarrow{\delta} H^k(X) \xrightarrow{i_\sharp} H^k(U_1) \oplus H^k(U)$$

by Mayer–Victoris. Since the right hand and left hand terms are finite dimensional it follows from Lemma 5.3.12 that the middle term is also finite dimensional. □

The proof works practically verbatim for compactly supported cohomology. For $N = 1$

$$H_c^k(X) = H_c^k(U_1) = H_c^k(\mathbb{R}^n)$$

so all the cohomology groups of $H^k(X)$ are finite in this case, and the induction "$N-1$" $\Rightarrow$ "N" follows from the exact sequence

$$H_c^k(U_1) \oplus H_c^k(U) \xrightarrow{j_\sharp} H_c^k(X) \xrightarrow{\delta} H_c^{k+1}(U_1 \cap U)\,.$$

Remark 5.3.13. *A careful analysis of the proof above shows that the dimensions of the $H^k(X)$'s are determined by the intersection properties of the U_i's, i.e., by the list of multi-indices, I, for which th intersections (5.3.1) are non-empty.*

This collection of multi-indices is called the *nerve* of the cover, $\mathbb{U} = \{U_i\,,\, i = 1, \dots, N\}$, and this remark suggests that there should be a cohomology theory which has as input the nerve of $\mathbb{U}$ and as output cohomology groups which are isomorphic to the DeRham cohomology groups. Such a theory does exist and a nice account of it can be found in Frank Warner's book, "Foundations of Differentiable Manifolds and Lie Groups". (See the section on Čech cohomology in Chapter 5.)

Exercises.

1. Let U be a bounded open subset of $\mathbb{R}^n$. A continuous function

$$\psi : U \to [0, \infty)$$

is called an exhaustion function if it is proper as a map of U into $[0, \infty)$; i.e., if, for every $a > 0$, $\psi^{-1}([0, a])$ is compact. For $x \in U$ let

$$d(x) = \inf \{|x - y| , \quad y \in \mathbb{R}^n - U\} ,$$

i.e., let $d(x)$ be the "distance" from x to the boundary of U. Show that $d(x) > 0$ and that $d(x)$ is continuous as a function of x. Conclude that $\psi_0 = 1/d$ is an exhaustion function.

2. Show that there exists a $\mathcal{C}^\infty$ exhaustion function, $\varphi_0 : U \to [0, \infty)$, with the property $\varphi_0 \geq \psi_0^2$ where ψ_0 is the exhaustion function in exercise 1.

Hints: For $i = 2, 3, \ldots$ let

$$C_i = \left\{ x \in U , \quad \frac{1}{i} \leq d(x) \leq \frac{1}{i-1} \right\}$$

and

$$U_i = \left\{ x \in U , \quad \frac{1}{i+1} < d(x) < \frac{1}{i-2} \right\} .$$

Let $\rho_i \in \mathcal{C}_0^\infty(U_i)$, $\rho_i \geq 0$, be a "bump" function which is identically one on C_i and let $\varphi_0 = \sum i^2 \rho_i + 1$.

3. Let U be a bounded open convex subset of $\mathbb{R}^n$ containing the origin. Show that there exists an exhaustion function

$$\psi : U \to \mathbb{R} , \quad \psi(0) = 1 ,$$

having the property that ψ is a *monotonically increasing* function of t along the ray, tx, $0 \leq t \leq 1$, for all points, x, in U. *Hints:*

(a) Let $\rho(x)$, $0 \leq \rho(x) \leq 1$, be a $\mathcal{C}^\infty$ function which is one outside a small neighborhood of the origin in U and is zero in a still smaller neighborhood of the origin. Modify the function, φ_0, in the previous exercise by setting $\varphi(x) = \rho(x)\varphi_0(x)$ and let

$$\psi(x) = \int_0^1 \varphi(sx)\frac{ds}{s} + 1\,.$$

Show that for $0 \leq t \leq 1$

$$\frac{d\psi}{dt}(tx) = \varphi(tx)/t \tag{5.3.5}$$

and conclude from (5.3.4) that ψ is monotonically increasing along the ray, tx, $0 \leq t \leq 1$.

(b) Show that for $0 < \epsilon < 1$,

$$\psi(x) \geq \epsilon\varphi(y)$$

where y is a point on the ray, tx, $0 \leq t \leq 1$ a distance less than $\epsilon|x|$ from X.

(c) Show that there exist constants, C_0 and C_1, $C_1 > 0$ such that

$$\psi(x) = \frac{C_1}{d(x)} + C_0\,.$$

Sub-hint: In part (b) take ϵ to be equal to $\frac{1}{2}\,d(x)/|x|$.

4. Show that every bounded, open convex subset, U, of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. *Hints:*

(a) Let $\psi(x)$ be the exhaustion function constructed in exercise 3 and let

$$f : U \to \mathbb{R}^n$$

be the map: $f(x) = \psi(x)x$. Show that this map is a bijective map of U onto $\mathbb{R}^n$.

(b) Show that for $x \in U$ and $\mathrm{v} \in \mathbb{R}^n$

$$(df)_x\mathrm{v} = \psi(x)\mathrm{v} + \,d\psi_x(\mathrm{v})x$$

and conclude that df_x is bijective at x, i.e., that f is locally a diffeomorphism of a neighborhood of x in U onto a neighborhood of $f(x)$ in $\mathbb{R}^n$.

(c) Putting (a) and (b) together show that f is a diffeomorphism of U onto $\mathbb{R}^n$.

5. Let $U \subseteq \mathbb{R}$ be the union of the open intervals, $k < x < k+1$, k an integer. Show that U *doesn't* have finite topology.

6. Let $V \subseteq \mathbb{R}^2$ be the open set obtained by deleting from $\mathbb{R}^2$ the points, $p_n = (0, n)$, n an integer. Show that V *doesn't* have finite topology. *Hint:* Let γ_n be a circle of radius $\frac{1}{2}$ centered about the point p_n. Using exercises 16–17 of §2.1 show that there exists a closed $\mathcal{C}^\infty$-one-form, ω_n on V with the property that $\int_{\gamma_n} \omega_n = 1$ and $\int_{\gamma_m} \omega_n = 0$ for $m \neq n$.

7. Let X be an n-dimensional manifold and $\mathbb{U} = \{U_i\,,\ i = 1, 2\}$ a good cover of X. What are the cohomology groups of X if the nerve of this cover is

(a) $\{1\}$, $\{2\}$

(b) $\{1\}$, $\{2\}$, $\{1, 2\}$?

8. Let X be an n-dimensional manifold and $\mathbb{U} = \{U_i\,,\ i = 1, 2, 3,\}$ a good cover of X. What are the cohomology groups of X if the nerve of this cover is

(a) $\{1\}$, $\{2\}$, $\{3\}$

(b) $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$

(c) $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$

(d) $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$

(e) $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$?

9. Let S^1 be the unit circle in $\mathbb{R}^3$ parametrized by arc length: $(x, y) = (\cos\theta, \sin\theta)$. Let U_1 be the set: $0 < \theta < \frac{2\pi}{3}$, U_2 the set: $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, and U_3 the set: $-\frac{2\pi}{3} < \theta < \frac{\pi}{3}$.

(a) Show that the U_i's are a good cover of S^1.

(b) Using the previous exercise compute the cohomology groups of S^1.

10. Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. Show that the sets

$$U_i = \{(x_1, x_2, x_3) \in S^2\,,\ x_i > 0\}$$

$i = 1, 2, 3$ and

$$U_i = \{(x_1, x_2, x_3) \in S^2\,,\ x_{i-3} < 0\}\,,$$

$i = 4, 5, 6$, are a good cover of S^2. What is the nerve of this cover?

11. Let X and Y be manifolds. Show that if they both have finite topology, their product, $X \times Y$, does as well.

12. (a) Let X be a manifold and let U_i, $i = 1, \dots, N$, be a good cover of X. Show that $U_i \times \mathbb{R}$, $i = 1, \dots, N$, is a good cover of $X \times \mathbb{R}$ and that the nerves of these two covers are the same.

(b) By Remark 5.3.13,

$$H^k(X \times \mathbb{R}) = H^k(X)\,.$$

Verify this directly using homotopy techniques.

(c) More generally, show that for all $\ell > 0$

$$H^k(X \times \mathbb{R}^\ell) = H^k(X) \tag{5.3.6}$$

(i) by concluding that this has to be the case in view of the Remark 5.3.13 and

(ii) by proving this directly using homotopy techniques.

5.4 Poincaré duality

In this chapter we've been studying two kinds of cohomology groups: the ordinary DeRham cohomology groups, H^k, and the compactly supported DeRham cohomology groups, H^k_c. It turns out that these groups are closely related. In fact if X is a connected, oriented n-dimensional manifold and has finite topology, $H^{n-k}_c(X)$ is the vector space dual of $H^k(X)$. We'll give a proof of this later in this section, however, before we do we'll need to review some basic linear algebra. Given two finite dimensional vector space, V and W, a *bilinear pairing* between V and W is a map

$$B : V \times W \to \mathbb{R} \tag{5.4.1}$$

which is linear in each of its factors. In other words, for fixed $w \in W$, the map

$$\ell_w : V \to \mathbb{R}\,, \quad \mathrm{v} \to B(\mathrm{v}, w) \tag{5.4.2}$$

is linear, and for $\mathrm{v} \in V$, the map

$$\ell_{\mathrm{v}} : W \to \mathbb{R}\,, \quad w \to B(\mathrm{v}, w) \tag{5.4.3}$$

is linear. Therefore, from the pairing (5.4.1) one gets a map

$$L_B : W \to V^*\,, \quad w \to \ell_w \tag{5.4.4}$$

and since $\ell_{w_1} + \ell_{w_2}(\mathrm{v}) = B(\mathrm{v}, w_1 + w_2) = \ell_{w_1+w_2}(\mathrm{v})$, this map is linear. We'll say that (5.4.1) is a non-singular pairing if (5.4.4) is bijective. Notice, by the way, that the roles of V and W can be reversed in this definition. Letting $B^\sharp(w, \mathrm{v}) = B(\mathrm{v}, w)$ we get an analogous linear map

$$L_{B^\sharp} : V \to W^* \tag{5.4.5}$$

and in fact

$$(L_{B^\sharp}(\mathrm{v}))(w) = (L_B(w))(\mathrm{v}) = B(\mathrm{v}, w)\,. \tag{5.4.6}$$

Thus if

$$\mu : V \to (V^*)^* \tag{5.4.7}$$

is the canonical identification of V with $(V^*)^*$ given by the recipe

$$\mu(\mathrm{v})(\ell) = \ell(\mathrm{v})$$

for $\mathrm{v} \in V$ and $\ell \in V^*$, we can rewrite (5.4.6) more suggestively in the form

$$L_{B^\sharp} = (L_B)^* \mu \tag{5.4.8}$$

i.e., L_B and $L_{B^\sharp}$ are just the transposes of each other. In particular L_B is bijective if and only if $L_{B^\sharp}$ is bijective.

Let's now apply these remarks to DeRham theory. Let X be a connected, oriented n-dimensional manifold. If X has finite topology the vector spaces, $H_c^{n-k}(X)$ and $H^k(X)$ are both finite dimensional. We will show that there is a natural bilinear pairing between these spaces, and hence by the discussion above, a natural linear mapping of $H^k(X)$ into the vector space dual of $H_c^{n-1}(X)$. To see this let c_1 be a cohomology class in $H_c^{n-k}(X)$ and c_2 a cohomology class in $H^k(X)$. Then by (5.1.42) their product, $c_1 \cdot c_2$, is an element of $H_c^n(X)$, and so by (5.1.8) we can define a pairing between c_1 and c_2 by setting

$$B(c_1, c_2) = I_X(c_1 \cdot c_2)\,. \tag{5.4.9}$$

Notice that if $\omega_1 \in \Omega_c^{n-k}(X)$ and $\omega_2 \in \Omega^k(X)$ are closed forms representing the cohomology classes, c_1 and c_2, then by (5.1.42) this pairing is given by the integral

$$B(c_1, c_2) = \int_X \omega_1 \wedge \omega_2\,. \tag{5.4.10}$$

We'll next show that this bilinear pairing is non-singular in one important special case:

Proposition 5.4.1. *If X is diffeomorphic to $\mathbb{R}^n$ the pairing defined by (5.4.9) is non-singular.*

Proof. To verify this there is very little to check. The vector spaces, $H^k(\mathbb{R}^n)$ and $H_c^{n-k}(\mathbb{R}^n)$ are zero except for $k = 0$, so all we have to check is that the pairing

$$H_c^n(X) \times H^0(X) \to \mathbb{R}$$

is non-singular. To see this recall that every compactly supported n-form is closed and that the only closed zero-forms are the constant functions, so at the level of forms, the pairing (5.4.9) is just the pairing

$$(\omega, c) \in \Omega^n(X) \times \mathbb{R} \to c \int_X \omega\,,$$

and this is zero if and only if c is zero or ω is in $d\Omega_c^{n-1}(X)$. Thus at the level of cohomology this pairing is non-singular.

□

We will now show how to prove this result in general.

Theorem 5.4.2 (Poincaré duality.)**.** *Let X be an oriented, connected n-dimensional manifold having finite topology. Then the pairing (5.4.9) is non-singular.*

The proof of this will be very similar in spirit to the proof that we gave in the last section to show that if X has finite topology its DeRham cohomology groups are finite dimensional. Like that proof, it involves Mayer–Victoris plus some elementary diagram-chasing. The "diagram-chasing" part of the proof consists of the following two lemmas.

Lemma 5.4.3. *Let V_1, V_2 and V_3 be finite dimensional vector spaces, and let $V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3$ be an exact sequence of linear mappings. Then the sequence of transpose maps*

$$V_3^* \xrightarrow{\beta^*} V_2^* \xrightarrow{\alpha^*} V_1$$

is exact.

Proof. Given a vector subspace, W_2, of V_2, let

$$W_2^\perp = \{\ell \in V_2^* \,;\, \ell(w) = 0 \text{ for } w \in W\}\,.$$

We'll leave for you to check that if W_2 is the kernel of β, then $W_2^\perp$ is the image of β^* and that if W_2 is the image of α, $W_2^\perp$ is the kernel of α^*. Hence if $\operatorname{Ker}\beta = \operatorname{Image}\alpha$, $\operatorname{Image}\beta^* = \operatorname{kernel}\alpha^*$.

□

Lemma 5.4.4 (the five lemma)**.** *Let the diagram below be a commutative diagram with the properties:*

(i) All the vector spaces are finite dimensional.

(ii) The two rows are exact.

(iii) The linear maps, γ_i, $i = 1, 2, 4, 5$ are bijections.

Then the map, γ_3, is a bijection.

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\uparrow{\scriptstyle \gamma_1} & & \uparrow{\scriptstyle \gamma_2} & & \uparrow{\scriptstyle \gamma_3} & & \uparrow{\scriptstyle \gamma_4} & & \uparrow{\scriptstyle \gamma_5} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5\,.
\end{array}$$

Proof. We'll show that γ_3 is surjective. Given $a_3 \in A_3$ there exists a $b_4 \in B_4$ such that $\gamma_4(b_4) = \alpha_3(a_3)$ since γ_4 is bijective. Moreover, $\gamma_5(\beta_4(b_4)) = \alpha_4(\alpha_3(a_3)) = 0$, by the exactness of the top row.

Therefore, since γ_5 is bijective, $\beta_4(b_4) = 0$, so by the exactness of the bottom row $b_4 = \beta_3(b_3)$ for some $b_3 \in B_3$, and hence

$$\alpha_3(\gamma_3(b_3)) = \gamma_4(\beta_3(b_3)) = \gamma_4(b_4) = \alpha_3(a_3)\,.$$

Thus $\alpha_3(a_3 - \gamma_3(b_3)) = 0$, so by the exactness of the top row

$$a_3 - \gamma_3(b_3) = \alpha_2(a_2)$$

for some $a_2 \in A_2$. Hence by the bijectivity of γ_2 there exists a $b_2 \in B_2$ with $a_2 = \gamma_2(b_2)$, and hence

$$a_3 - \gamma_3(b_3) = \alpha_2(a_2) = \alpha_2(\gamma_2(b_2)) = \gamma_3(\beta_2(b_2))\,.$$

Thus finally

$$a_3 = \gamma_3(b_3 + \beta_2(b_2))\,.$$

Since a_3 was any element of A_3 this proves the surjectivity of γ_3.

One can prove the injectivity of γ_3 by a similar diagram-chasing argument, but one can also prove this with less duplication of effort by taking the transposes of all the arrows in Figure 5.4.1 and noting that the same argument as above proves the surjectivity of $\gamma_3^* : A_3^* \to B_3^*$.

□

To prove Theorem 5.4.2 we apply these lemmas to the diagram below. In this diagram U_1 and U_2 are open subsets of X, M is $U_1 \cup U_2$ and the vertical arrows are the mappings defined by the pairing (5.4.9). We will leave for you to check that this is a commutative diagram "up to sign". (To make it commutative one has to replace some of the vertical arrows, γ, by their negatives: $-\gamma$.) This is easy to check except for the commutative square on the extreme left. To check that this square commutes, some serious diagram-chasing is required.

$$\begin{array}{ccccccccc}
\to & H^{n-(k-1)}(M) & \to & H^{n-k}(U_1 \cap U_2)^* & \to & H^{n-k}(U_1)^* \oplus H^{n-k}(U_2)^* & \to & H^{n-k}(M)^* & \to \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\to & H_c^{k-1}(M) & \to & H_c^k(U_1 \cap U_2) & \to & H_c^k(U_1) \oplus H_c^k(U_2) & \to & H_c^k(M) & \to
\end{array}$$

Figure 5.4.2

By Mayer–Victoris the bottom row of this figure is exact and by Mayer–Victoris and Lemma 5.4.3 the top row of this figure is exact. hence we can apply the "five lemma" to Figure 5.4.2 and conclude:

Lemma 5.4.5. *If the maps*

$$H^k(U) \to H_c^{n-k}(U)^* \tag{5.4.11}$$

defined by the pairing (5.4.9) are bijective for U_1, U_2 *and* $U_1 \cap U_2$*, they are also bijective for* $M = U_1 \cup U_2$*.*

Thus to prove Theorem 5.4.2 we can argue by induction as in § 5.3. Let $U_1, U_2, \ldots, U_N$ be a good cover of X. If $N = 1$, then $X = U_1$ and, hence, since U_1 is diffeomorphic to $\mathbb{R}^n$, the map (5.4.12) is bijective by Proposition 5.4.1. Now let's assume the theorem is true for manifolds involving good covers by k open sets where k is less than N. Let $U' = U_1 \cup \cdots \cup U_{N-1}$ and $U'' = U_N$. Since

$$U' \cap U'' = U_1 \cap U_N \cup \cdots \cup U_{N-1} \cap U_N$$

it can be covered by a good cover by k open sets, $k < N$, and hence the hypotheses of the lemma are true for U', U'' and $U' \cap U''$. Thus the lemma says that (5.4.12) is bijective for the union, X, of U' and U''. □

Exercises.

1. (The "push-forward" operation in DeRham cohomology.) Let X be an m-dimensional manifold, Y an n-dimensional manifold and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Suppose that both of these manifolds are oriented and connected and have finite topology. Show that there exists a unique linear map

$$f_\sharp : H_c^{m-k}(X) \to H_c^{n-k}(Y) \tag{5.4.12}$$

with the property

$$B_Y(f_\sharp c_1, c_2) = B_X(c_1, f^\sharp c_2) \tag{5.4.13}$$

for all $c_1 \in H_c^{m-k}(X)$ and $c_2 \in H^k(Y)$. (In this formula B_X is the bilinear pairing (5.4.9) on X and B_Y is the bilinear pairing (5.4.9) on Y.)

2. Suppose that the map, f, in exercise 1 is proper. Show that there exists a unique linear map

$$f_\sharp : H^{m-k}(X) \to H^{n-k}(Y) \tag{5.4.14}$$

with the property

$$B_Y(c_1, f_\sharp c_2) = (-1)^{k(m-n)} B_X(f^\sharp c_1, c_2) \tag{5.4.15}$$

for all $c_1 \in H^k_c(Y)$ and $c_2 \in H^{m-k}(X)$, and show that, if X and Y are compact, this mapping is the same as the mapping, $f_\sharp$, in exercise 1.

3. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \times \mathbb{R} \to U$ be the projection, $f(x,t) = x$. Show that there is a unique linear mapping

$$f_* : \Omega^{k+1}_c(U \times \mathbb{R}) \to \Omega^k_c(U) \tag{5.4.16}$$

with the property

$$\int_U f_*\mu \wedge \nu = \int_{U\times\mathbb{R}} \mu \wedge f^*\nu \tag{5.4.17}$$

for all $\mu \in \Omega^{k+1}_c(U \times \mathbb{R})$ and $\nu \in \Omega^{n-k}(U)$.

Hint: Let $x_1, \dots, x_n$ and t be the standard coordinate functions on $\mathbb{R}^n \times \mathbb{R}$. By §2.2, exercise 5 every $(k+1)$-form, $\omega \in \Omega^{k+1}_c(U \times \mathbb{R})$ can be written uniquely in "reduced form" as a sum

$$\omega = \sum f_I \, dt \wedge dx_I + \sum g_J \, dx_J$$

over multi-indices, I and J, which are strictly increasing. Let

$$f_*\omega = \sum_I \left(\int_{\mathbb{R}} f_I(x,t)\, dt \right) dx_I \,. \tag{5.4.18}$$

4. Show that the mapping, f_*, in exercise 3 satisfies $f_*\, d\omega = df_*\omega$.

5. Show that if ω is a closed compactly supported $k+1$-form on $U \times \mathbb{R}$ then

$$[f_*\omega] = f_\sharp[\omega] \tag{5.4.19}$$

where $f_\sharp$ is the mapping (5.4.13) and f_* the mapping (5.4.17).

6. (a) Let U be an open subset of $\mathbb{R}^n$ and let $f : U \times \mathbb{R}^\ell \to U$ be the projection, $f(x,t) = x$. Show that there is a unique linear mapping

$$f_* : \Omega_c^{k+\ell}(U \times \mathbb{R}^\ell) \to \Omega_c^k(U) \tag{5.4.20}$$

with the property

$$\int_U f_*\mu \wedge \nu = \int_{U\times\mathbb{R}^\ell} \mu \wedge f^*\nu \tag{5.4.21}$$

for all $\mu \in \Omega_c^{k+\ell}(U \times \mathbb{R}^\ell)$ and $\nu \in \Omega^{n-k}(U)$.

Hint: Exercise 3 plus induction on ℓ.

(b) Show that for $\omega \in \Omega_c^{k+\ell}(U \times \mathbb{R}^\ell)$

$$df_*\omega = f_*\, d\omega\,.$$

(c) Show that if ω is a closed, compactly supported $k+\ell$-form on $X \times \mathbb{R}^\ell$

$$f_\sharp[\omega] = [f_*\omega] \tag{5.4.22}$$

where $f_\sharp : H_c^{k+\ell}(U \times \mathbb{R}^\ell) \to H_c^k(U)$ is the map (5.4.13).

7. Let X be an n-dimensional manifold and Y an m-dimensional manifold. Assume X and Y are compact, oriented and connected, and orient $X \times Y$ by giving it its natural product orientation. Let

$$f : X \times Y \to Y$$

be the projection map, $f(x,y) = y$. Given

$$\omega \in \Omega^m(X \times Y)$$

and $p \in Y$, let

$$f_*\omega(p) = \int_X \iota_p^*\omega \tag{5.4.23}$$

where $\iota_p : X \to X \times Y$ is the inclusion map, $\iota_p(x) = (x,p)$.

(a) Show that the function $f_*\omega$ defined by (5.5.24) is $\mathcal{C}^\infty$, i.e., is in $\Omega^0(Y)$.

(b) Show that if ω is closed this function is constant.

(c) Show that if ω is closed

$$[f_*\omega] = f_\sharp[\omega]$$

where $f_\sharp : H^n(X \times Y) \to H^0(Y)$ is the map (5.4.13).

8. (a) Let X be an n-dimensional manifold which is compact, connected and oriented. Combining Poincaré duality with exercise 12 in § 5.3 show that

$$H_c^{k+\ell}(X \times \mathbb{R}^\ell) = H_c^k(X)\,.$$

(b) Show, moreover, that if $f : X \times \mathbb{R}^\ell \to X$ is the projection, $f(x,a) = x$, then

$$f_\sharp : H_c^{k+\ell}(X \times \mathbb{R}^\ell) \to H_c^k(X)$$

is a bijection.

9. Let X and Y be as in exercise 1. Show that the push-forward operation (5.4.13) satisfies

$$f_\sharp; (c_1 \cdot f^\sharp c_2) = f_\sharp c_1 \cdot c_2$$

for $c_1 \in H_c^k(X)$ and $c_2 \in H^\ell(Y)$.

5.5 Thom classes and intersection theory

Let X be a connected, oriented n-dimensional manifold. If X has finite topology its cohomology groups are finite dimensional, and since the bilinear pairing, B, defined by (5.4.9) is non-singular we get from this pairing bijective linear maps

(5.5.1) $$L_B : H_c^{n-k}(X) \to H^k(X)^*$$

and

(5.5.2) $$L_B^* : H^{n-k}(X) \to H_c^k(X)^*\,.$$

In particular, if $\ell : H^k(X) \to \mathbb{R}$ is a linear function (i.e., an element of $H^k(X)^*$), then by (5.5.1) we can convert ℓ into a cohomology class

(5.5.3) $$L_B^{-1}(\ell) \in H_c^{n-k}(X)\,,$$

and similarly if $\ell_c : H^k_c(X) \to \mathbb{R}$ is a linear function, we can convert it by (5.5.2) into a cohomology class

$$(L_B^*)^{-1}(\ell) \in H^{n-k}(X)\,. \tag{5.5.4}$$

One way that linear functions like this arise in practice is by integrating forms over submanifolds of X. Namely let Y be a closed, oriented k dimensional submanifold of X. Since Y is oriented, we have by (5.1.8) an integration operation in cohomology

$$I_Y : H^k_c(Y) \to \mathbb{R}\,,$$

and since Y is closed the inclusion map, ι_Y, of Y into X is proper, so we get from it a pull-back operation on cohomology

$$(\iota_Y)^\sharp : H^k_c(X) \to H^k_c(Y)$$

and by composing these two maps, we get a linear map, $\ell_Y = I_Y \circ (\iota_Y)^\sharp$, of $H^k_c(X)$ into $\mathbb{R}$. The cohomology class

$$T_Y = L_B^{-1}(\ell_Y) \in H^k_c(X) \tag{5.5.5}$$

associated with ℓ_Y is called the *Thom class* of the manifold, Y and has the defining property

$$B(T_Y, c) = I_Y(\iota_Y^\sharp c) \tag{5.5.6}$$

for $c \in H^k_c(X)$. Let's see what this defining property looks like at the level of forms. Let $\tau_Y \in \Omega^{n-k}(X)$ be a closed k-form representing T_Y. Then by (5.4.9), the formula (5.5.6), for $c = [\omega]$, becomes the integral formula

$$\int_X \tau_Y \wedge \omega = \int_Y \iota_Y^* \omega\,. \tag{5.5.7}$$

In other words, for every closed form, $\omega \in \Omega^{n-k}_c(X)$ the integral of ω over Y is equal to the integral over X of $\tau_Y \wedge \omega$. A closed form, τ_Y, with this "reproducing" property is called a *Thom form* for Y. Note that if we add to τ_Y an exact $(n-k)$-form, $\mu \in d\Omega^{n-k-1}(X)$, we get another representative, $\tau_Y + \mu$, of the cohomology class, T_Y, and hence another form with this reproducing property. Also, since the formula (5.5.7) is a direct translation into form language of the formula (5.5.6) any closed $(n-k)$-form, τ_Y, with the reproducing property (5.5.7) is a representative of the cohomology class, T_Y.

These remarks make sense as well for compactly supported cohomology. Suppose Y is compact. Then from the inclusion map we get a pull-back map

$$(\iota_Y)^\sharp : H^k(X) \to H^k(Y)$$

and since Y is compact, the integration operation, I_Y, is a map of $H^k(Y)$ into $\mathbb{R}$, so the composition of these two operations is a map,

$$\ell_Y : H^k(X) \to \mathbb{R}$$

which by (5.5.3) gets converted into a cohomology class

$$T_Y = L_B^{-1}(\ell_Y) \in H_c^{n-k}(X)\,.$$

Moreover, if $\tau_Y \in \Omega_c^{n-k}(X)$ is a closed form, it represents this cohomology class if and only if it has the reproducing property

$$\int_X \tau_Y \wedge \omega = \int_Y \iota_Y^* \omega \tag{5.5.8}$$

for closed forms, ω, in $\Omega^{n-k}(X)$. (There's a subtle difference, however, between formula (5.5.7) and formula (5.5.8). In (5.5.7) ω has to be closed *and* compactly supported and in (5.5.8) it just has to be closed.)

As above we have a lot of latitude in our choice of τ_Y: we can add to it any element of $d\Omega_c^{n-k-1}(X)$. One consequence of this is the following.

Theorem 5.5.1. *Given a neighborhood, U, of Y in X there exists a closed form, $\tau_Y \in \Omega_c^{n-k}(U)$, with the reproducing property*

$$\int_U \tau_Y \wedge \omega = \int_Y \iota_Y^* \omega \tag{5.5.9}$$

for closed forms, $\omega \in \Omega^k(U)$.

Hence in particular, τ_Y has the reproducing property (5.5.8) for closed forms, $\omega \in \Omega^{n-k}(X)$. This result shows that the Thom form, τ_Y, can be chosen to have support in an *arbitrarily small neighborhood* of Y. To prove Theorem 5.5.1 we note that by Theorem 5.3.8 we can assume that U has finite topology and hence, in our definition of τ_Y, we can replace the manifold, X, by the open submanifold,

U. This gives us a Thom form, τ_Y, with support in U and with the reproducing property (5.5.9) for closed forms $\omega \in \Omega^{n-k}(U)$.

□

Let's see what Thom forms actually look like in concrete examples. Suppose Y is defined globally by a system of ℓ independent equations, i.e., suppose there exists an open neighborhood, $\mathcal{O}$, of Y in X, a $\mathcal{C}^\infty$ map, $f : \mathcal{O} \to \mathbb{R}^\ell$, and a bounded open convex neighborhood, V, of the origin in $\mathbb{R}^n$ such that

$$\text{(5.5.10)} \qquad \begin{aligned} &\text{(i)} \quad \text{The origin is a regular value of } f\,. \\ &\text{(ii)} \quad f^{-1}(\bar{V}) \text{ is closed in } X\,. \\ &\text{(iii)} \quad Y = f^{-1}(0)\,. \end{aligned}$$

Then by (i) and (iii) Y is a closed submanifold of $\mathcal{O}$ and by (ii) it's a closed submanifold of X. Moreover, it has a natural orientation: For every $p \in Y$ the map

$$df_p : T_pX \to T_0\mathbb{R}^\ell$$

is surjective, and its kernel is T_pY, so from the standard orientation of $T_0\mathbb{R}^\ell$ one gets an orientation of the quotient space,

$$T_pX/T_pY\,,$$

and hence since T_pX is oriented, one gets, by Theorem 1.9.4, an orientation on T_pY. (See §4.4, example 2.) Now let μ be an element of $\Omega^\ell_c(X)$. Then $f^*\mu$ is supported in $f^{-1}(\bar{V})$ and hence by property (ii) of (5.5.10) we can extend it to X by setting it equal to zero outside $\mathcal{O}$. We will prove

Theorem 5.5.2. *If*

$$\text{(5.5.11)} \qquad \int_V \mu = 1\,,$$

$f^*\mu$ is a Thom form for Y.

To prove this we'll first prove that if $f^*\mu$ has property (5.5.7) for some choice of μ it has this property for every choice of μ.

Lemma 5.5.3. *Let μ_1 and μ_2 be forms in $\Omega^\ell_c(V)$ with the property (5.5.11). Then for every closed k-form, $\nu \in \Omega^k_c(X)$*

$$\int_X f^*\mu_1 \wedge \nu = \int_X f^*\mu_2 \wedge \nu\,.$$

Proof. By Theorem 3.2.1, $\mu_1 - \mu_2 = d\beta$ for some $\beta \in \Omega_c^{\ell-1}(V)$, hence, since $d\nu = 0$

$$(f^*\mu_1 - f^*\mu_2) \wedge \nu = df^*\beta \wedge \nu = d(f^*\beta \wedge \nu)\,.$$

Therefore, by Stokes theorem, the integral over X of the expression on the left is zero.

□

Now suppose $\mu = \rho(x_1, \dots, x_\ell)\, dx_1 \wedge \cdots \wedge dx_\ell$, for ρ in $\mathcal{C}_0^\infty(V)$. For $t \leq 1$ let

$$\mu_t = t^\ell \rho\left(\frac{x_1}{t}, \cdots, \frac{x_\ell}{t}\right) dx_1 \wedge \cdots dx_\ell\,. \tag{5.5.12}$$

This form is supported in the convex set, tV, so by Lemma 5.5.3

$$\int_X f^*\mu_t \wedge \nu = \int_X f^*\mu \wedge \nu \tag{5.5.13}$$

for all closed forms $\nu \in \Omega_c^k(X)$. Hence to prove that $f^*\mu$ has the property (5.5.7) it suffices to prove

$$\mathrm{Lim}_{t\to 0} \int f^*\mu_t \wedge \nu = \int_Y \iota_Y^*\nu\,. \tag{5.5.14}$$

We'll prove this by proving a stronger result.

Lemma 5.5.4. *The assertion (5.5.14) is true for every k-form $\nu \in \Omega_c^k(X)$.*

Proof. The canonical form theorem for submersions (see Theorem 4.3.6) says that for every $p \in Y$ there exists a neighborhood U_p of p in Y, a neighborhood, W of 0 in $\mathbb{R}^n$, and an orientation preserving diffeomorphism $\psi : (W, 0) \to (U_p, p)$ such that

$$f \circ \psi = \pi \tag{5.5.15}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^\ell$ is the canonical submersion, $\pi(x_1, \dots, x_n) = (x_1, \dots, x_\ell)$. Let $\mathbb{U}$ be the cover of $\mathcal{O}$ by the open sets, $\mathcal{O} - Y$ and the U_p's. Choosing a partition of unity subordinate to this cover it suffices to verify (5.5.14) for ν in $\Omega_c^k(\mathcal{O} - Y)$ and ν in $\Omega_c^k(U_p)$. Let's first suppose ν is in $\Omega_c^k(\mathcal{O} - Y)$. Then $f(\operatorname{supp}\nu)$ is a compact subset of $\mathbb{R}^\ell - \{0\}$ and hence for t small $f(\operatorname{supp}\nu)$ is disjoint from tV, and both sides of (5.5.14) are zero. Next suppose that ν is in $\Omega^k_c(U_p)$. Then $\psi^*\nu$ is a compactly supported k-form on W so we can write it as a sum

$$\psi^*\nu = \sum h_I(x)\, dx_I\,, \quad h_I \in \mathcal{C}^\infty_0(W)$$

the I's being strictly increasing multi-indices of length k. Let $I_0 = (\ell+1, \ell_2+2, \dots, n)$. Then

$$\pi^*\mu_t \wedge \psi^*\nu = t^\ell \rho(\frac{x_1}{t}\,, \cdots\,, \frac{x_\ell}{t}) h_{I_0}(x_1, \dots, x_n)\, dx_r \wedge \cdots dx_n \tag{5.5.16}$$

and by (5.5.15)

$$\psi^*(f^*\mu_t \wedge \nu) = \pi^*\mu_t \wedge \psi^*\nu$$

and hence since ψ is orientation preserving

$$\begin{aligned}\int_{U_p} f^*\mu_t \wedge \nu &= t^\ell \int_{\mathbb{R}^n} \rho\left(\frac{x_1}{t}\,, \cdots\,, \frac{x_\ell}{t}\right) h_{I_0}(x_1, \dots, x_n)\, dx \\ &= \int_{\mathbb{R}^n} \rho(x_1, \dots, x_\ell) h_{I_0}(tx_1, \dots, tx_\ell\,,\, x_{\ell+1}, \dots, x_n)\, dx\end{aligned}$$

and the limit of this expression as t tends to zero is

$$\int \rho(x_1, \dots, x_\ell) h_{I_0}(0, \dots, 0\,,\, x_{\ell+1}, \dots, x_n)\, dx_1 \dots\, dx_n$$

or

$$\int h_I(0, \dots, 0\,,\, x_{\ell+1}, \dots, x_n)\, dx_{\ell+1} \cdots\, dx_n\,. \tag{5.5.17}$$

This, however, is just the integral of $\psi^*\nu$ over the set $\pi^{-1}(0) \cap W$. By (5.5.14) ψ maps this set diffeomorphically onto $Y \cap U_p$ and by our recipe for orienting Y this diffeomorphism is an orientation-preserving diffeomorphism, so the integral (5.5.17) is equal to the integral of ν over Y.

□

We'll now describe some applications of Thom forms to *topological intersection theory*. Let Y and Z be closed, oriented submanifolds of X of dimensions k and ℓ where $k + \ell = n$, and let's assume one of them (say Z) is compact. We will show below how to define an "intersection number", $I(Y, Z)$, which on the one hand will be a topological invariant of Y and Z and on the other hand will actually count, with appropriate ±-signs, the number of points of intersection of Y and Z when they intersect non-tangentially. (Thus this notion is similar to the notion of "degree f" for a $\mathcal{C}^\infty$ mapping f. On the one hand "degree f" is a topological invariant of f. It's unchanged if we deform f by a homotopy. On the other hand if q is a regular value of f, "degree f" counts with appropriate ±-signs the number of points in the set, $f^{-1}(q)$.)

We'll first give the topological definition of this intersection number. This is by the formula

$$I(Y,Z) = B(T_Y, T_Z) \tag{5.5.18}$$

where $T_Y \in H^\ell(X)$ and $T_Z \in H^k_c(X)$ and B is the bilinear pairing (5.4.9). If $\tau_Y \in \Omega^\ell(X)$ and $\tau_Z \in \Omega^k_c(X)$ are Thom forms representing T_Y and T_Z, (5.5.18) can also be defined as the integral

$$I(Y,Z) = \int_X \tau_Y \wedge \tau_Z \tag{5.5.19}$$

or by (5.5.9), as the integral over Y,

$$I(Y,Z) = \int_Y \iota_Y^* \tau_Z \tag{5.5.20}$$

or, since $\tau_Y \wedge \tau_Z = (-1)^{k\ell} \tau_Z \wedge \tau_Y$, as the integral over Z

$$I(X,Y) = (-1)^{k\ell} \int_Z \iota_Z^* \tau_Y \,. \tag{5.5.21}$$

In particular

$$I(Y,Z) = (-1)^{k\ell} I(Z,Y). \tag{5.5.22}$$

As a test case for our declaring $I(Y,Z)$ to be the intersection number of Y and Z we will first prove:

Proposition 5.5.5. *If Y and Z* don't *intersect, then $I(Y,Z) = 0$.*

Proof. If Y and Z don't intersect then, since Y is closed, $U = X - Y$ is an open neighborhood of Z in X, therefore since Z is compact there exists by Theorem 5.5.1 a Thom form, τ_Z in $\Omega^\ell_c(U)$. Thus $\iota_Y^* \tau_Z = 0$, and so by (5.5.20) $I(Y,Z) = 0$.

□

We'll next indicate how one computes $I(Y,Z)$ when Y and Z intersect "non-tangentially", or, to use terminology more in current usage, when their intersection is *transversal*. Recall that at a point of intersection, $p \in Y \cap Z$, T_pY and T_pZ are vector subspaces of T_pX.

Definition 5.5.6. *Y and Z intersect* transversally *if for every $p \in Y \cap Z$, $T_pY \cap T_pZ = \{0\}$.*

Since $n = k + \ell = \dim T_pY + \dim T_pZ = \dim T_pX$, this condition is equivalent to

$$T_pX = T_pY \oplus T_pZ\,, \tag{5.5.23}$$

i.e., every vector, $u \in T_pX$, can be written uniquely as a sum, $u = \mathrm{v} + w$, with $\mathrm{v} \in T_pY$ and $w \in T_pZ$. Since X, Y and Z are oriented, their tangent spaces at p are oriented, and we'll say that these spaces are *compatibly oriented* if the orientations of the two sides of (5.5.23) agree. (In other words if $\mathrm{v}_1, \dots, \mathrm{v}_k$ is an oriented basis of T_pY and $w_1, \dots, w_\ell$ is an oriented basis of T_pZ, the n vectors, $\mathrm{v}_1, \dots, \mathrm{v}_k$, $w_1, \dots, w_\ell$, are an oriented basis of T_pX.) We will define the *local intersection number*, $I_p(Y,Z)$, of Y and Z at p to be equal to $+1$ if X, Y and Z are compatibly oriented at p and to be equal to -1 if they're not. With this notation we'll prove

Theorem 5.5.7. *If Y and Z intersect transversally then $Y \cap Z$ is a finite set and*

$$I(Y,Z) = \sum_{p \in Y \cap Z} I_p(Y,Z). \tag{5.5.24}$$

To prove this we first need to show that transverse intersections look nice locally.

Theorem 5.5.8. *If Y and Z intersect transversally, then for every $p \in Y \cap Z$, there exists an open neighborhood, V_p, of p in X, an open neighborhood, U_p, of the origin in $\mathbb{R}^n$ and an orientation preserving diffeomorphism*

$$\psi_p : V_p \to U_p$$

which maps $V_p \cap Y$ diffeomorphically onto the subset of U_p defined by the equations: $x_1 = \dots = x_\ell = 0$, and maps $V \cap Z$ onto the subset of U_p defined by the equations: $x_{\ell+1} = \dots = x_n = 0$.

Proof. Since this result is a local result, we can assume that X is $\mathbb{R}^n$ and hence by Theorem 4.2.7 that there exists a neighborhood, V_p, of p in $\mathbb{R}^n$ and submersions $f : (V_p, p) \to (\mathbb{R}^\ell, 0)$ and $g : (V_p, p) \to (\mathbb{R}^k, 0)$ with the properties

$$V_p \cap Y = f^{-1}(0) \tag{5.5.25}$$

and

$$\mathrm{v}_p \cap Z = g^{-1}(0)\,. \tag{5.5.26}$$

Moreover, by (4.3.4)

$$T_pY = (df_p)^{-1}(0)$$

and

$$T_pZ = (dg_p)^{-1}(0)\,.$$

Hence by (5.5.23), the equations

$$df_p(\mathrm{v}) = dg_p(\mathrm{v}) = 0 \tag{5.5.27}$$

for $\mathrm{v} \in T_pX$ imply that $\mathrm{v} = 0$. Now let $\psi_p : V_p \to \mathbb{R}^n$ be the map

$$(f, g) : V_p \to \mathbb{R}^\ell \times \mathbb{R}^k = \mathbb{R}^n\,.$$

Then by (5.5.27), $d\psi_p$ is bijective, therefore, shrinking V_p if necessary, we can assume that ψ_p maps V_p diffeomorphically onto a neighborhood, U_p, of the origin in $\mathbb{R}^n$, and hence by (5.5.25) and (5.5.26), ψ_p maps $V_p \cap Y$ onto the set: $x_1 = \cdots = x_\ell = 0$ and maps $V_p \cap Z$ onto the set: $x_{\ell+1} = \cdots = x_n = 0$. Finally, if ψ isn't orientation preserving, we can make it so by composing it with the involution, $(x_1, \ldots, x_n) \to (x_1, x_2, \ldots, x_{n-1}, -x_n)$.

□

From this result we deduce:

Theorem 5.5.9. *If Y and Z intersect transversally, their intersection is a finite set.*

Proof. By Theorem 5.5.8 the only point of intersection in V_p is p itself. Moreover, since Y is closed and Z is compact, $Y \cap Z$ is compact.

Therefore, since the V_p's cover $Y \cap Z$ we can extract a finite subcover by the Heine–Borel theorem. However, since no two V_p's cover the same point of $Y \cap Z$, this cover must already be a finite subcover.

□

We will now prove Theorem 5.5.7. Since Y is closed, the map, $\iota_Y : Y \to X$ is proper, so by Theorem 3.4.2 there exists a neighborhood, U, of Z in X such that $U \cap Y$ is contained in the union of the open sets, V_p, above. Moreover by Theorem 5.5.1 we can choose τ_Z to be supported in U and by Theorem 5.3.2 we can assume that U has finite topology, so we're reduced to proving the theorem with X replaced by U and Y replaced by $Y \cap U$. Let

$$\mathcal{O} = \left(\bigcup V_p \right) \cap U \,,$$

let

$$f : \mathcal{O} \to \mathbb{R}^\ell$$

be the map whose restriction to $V_p \cap U$ is $\pi \circ \psi_p$ where π is, as in (5.5.15), the canonical submersion of $\mathbb{R}^n$ onto $\mathbb{R}^\ell$, and finally let V be a bounded convex neighborhood of $\mathbb{R}^\ell$, whose closure is contained in the intersection of the open sets, $\pi \circ \psi_p(V_p \cap U)$. Then $f^{-1}(\bar{V})$ is a closed subset of U, so if we replace X by U and Y by $Y \cap U$, the data $(f, \mathcal{O}, V)$ satisfy the conditions (5.5.10). Thus to prove Theorem 5.5.7 it suffices by Theorem 5.5.2 to prove this theorem with

$$\tau_Y = \sigma_p(Y) f^* \mu$$

on $V_p \cap \mathcal{O}$ where $\sigma_p(Y) = +1$ or -1 depending on whether the orientation of $Y \cap V_p$ in Theorem 5.5.2 coincides with the given orientation of Y or not. Thus

$$\begin{aligned}
I(Y,Z) &= (-1)^{k\ell} I(Z,Y) \\
&= (-1)^{k\ell} \sum_p \sigma_p(Y) \int_Z \iota_Z^* f^* \mu \\
&= (-1)^{k\ell} \sum_p \sigma_p(Y) \int_Z \iota_Z^* \psi_p^* \pi^* \mu \\
&= \sum_p (-1)^{k\ell} \sigma_p(Y) \int_{Z \cap V_p} (\pi \circ \psi_p \circ \iota_Z)^* \mu \,.
\end{aligned}$$

But $\pi \circ \psi_p \circ \iota_Z$ maps an open neighborhood of p in $U_p \cap Z$ diffeomorphically onto V, and μ is compactly supported in V so by (5.5.11)

$$\int_{Z\cap U_p} (\pi \circ \psi_p \circ \iota_Z)^* \mu = \sigma_p(Z) \int_V \mu = \sigma_p(Z)$$

where $\sigma_p(Z) = +1$ or -1 depending on whether $\pi \circ \psi_p \circ \iota_Z$ is orientation preserving or not. Thus finally

$$I(Y, Z) = \sum (-1)^{k\ell} \sigma_p(Y) \sigma_p(Z) \,.$$

We will leave as an exercise the task of unraveling these orientations and showing that

$$(-1)^{k\ell} \sigma_p(Y) \sigma_p(Z) = I_p(Y, Z)$$

and hence that $I(Y, Z) = \sum_p I_p(Y, Z)$.

Exercises.

1. Let X be a connected, oriented n-dimensional manifold, W a connected, oriented ℓ-dimensional manifold, $f : X \to W$ a $\mathcal{C}^\infty$ map, and Y a closed submanifold of X of dimension $k = n - \ell$. Suppose Y is a "level set" of the map, f, i.e., suppose that q is a regular value of f and that $Y = f^{-1}(q)$. Show that if μ is in $\Omega_c^\ell(Z)$ and its integral over Z is 1, then one can orient Y so that $\tau_Y = f^*\mu$ is a Thom form for Y.

Hint: Theorem 5.5.2.

2. In exercise 1 show that if $Z \subseteq X$ is a compact oriented ℓ-dimensional submanifold of X then

$$I(Y, Z) = (-1)^{k\ell} \deg(f \circ \iota_Z) \,.$$

3. Let q_1 be another regular value of the map, $f : X \to W$, and let $Y_1 = f^{-1}(q)$. Show that

$$I(Y, Z) = I(Y_1, Z) \,.$$

4. (a) Show that if q is a regular value of the map, $f \circ \iota_Z : Z \to W$ then Z and Y intersect transversally.

(b) Show that this is an "if and only if" proposition: If Y and Z intersect transversally then q is a regular value of the map, $f \circ \iota_Z$.

5. Suppose q is a regular value of the map, $f \circ \iota_Z$. Show that p is in $Y \cap Z$ if and only if p is in the pre-image $(f \circ \iota_Z)^{-1}(q)$ of q and that

$$I_p(X, Y) = (-1)^{k\ell} \sigma_p$$

where σ_p is the orientation number of the map, $f \circ \iota_Z$, at p, i.e., $\sigma_p = 1$ if $f \circ \iota_Z$ is orientation-preserving at p and $\sigma_p = -1$ if $f \circ \iota_Z$ is orientation-reversving at p.

6. Suppose the map $f : X \to W$ is proper. Show that there exists a neighborhood, V, of q in W having the property that *all* points of V are regular values of f.

Hint: Since q is a regular value of f there exists, for every $p \in f^{-1}(q)$ a neighborhood, U_p of p, on which f is a submersion. Conclude, by Theorem 3.4.2, that there exists a neighborhood, V, of q with $f^{-1}(V) \subseteq \bigcup U_p$.

7. Show that in every neighborhood, V_1, of q in V there exists a point, q_1, whose pre-image

$$Y_1 = f^{-1}(q_1)$$

intersects Z transversally. (*Hint:* Exercise 4 plus Sard's theorem.) Conclude that one can "deform Y an arbitrarily small amount so that it intersects Z transversally".

8. (Intersection theory for mappings.) Let X be an oriented, connected n-dimensional manifold, Z a compact, oriented ℓ-dimensional submanifold, Y an oriented manifold of dimension $k = n - \ell$ and $f : Y \to X$ a proper $\mathcal{C}^\infty$ map. Define the intersection number of f with Z to be the integral

$$I(f, Z) = \int_Y f^* \tau_Z \,.$$

(a) Show that $I(f, Z)$ is a homotopy invariant of f, i.e., show that if $f_i : Y \to X$, $i = 0, 1$ are proper $\mathcal{C}^\infty$ maps and are properly homotopic, then

$$I(f_0, Z) = I(f_1, Z) \,.$$

(b) Show that if Y is a closed submanifold of X of dimension $k = n - \ell$ and $\iota_Y : Y \to X$ is the inclusion map

$$I(\iota_Y, Z) = I(Y, Z)\,.$$

9. (a) Let X be an oriented, connected n-dimensional manifold and let Z be a compact zero-dimensional submanifold consisting of a single point, $z_0 \in X$. Show that if μ is in $\Omega_c^n(X)$ then μ is a Thom form for Z if and only if its integral is 1.

(b) Let Y be an oriented n-dimensional manifold and $f : Y \to X$ a $\mathcal{C}^\infty$ map. Show that for $Z = \{z_0\}$ as in part a

$$I(f, Z) = \deg(f)\,.$$

5.6 The Lefshetz theorem

In this section we'll apply the intersection techniques that we developed in §5.5 to a concrete problem in dynamical systems: counting the number of fixed points of a differentiable mapping. The Brouwer fixed point theorem, which we discussed in §3.6, told us that a $\mathcal{C}^\infty$ map of the unit ball into itself has to have at least one fixed point. The Lefshetz theorem is a similar result for manifolds. It will tell us that a $\mathcal{C}^\infty$ map of a compact manifold into itself has to have a fixed point if a certain topological invariant of the map, its *global Lefshetz number*, is non-zero.

Before stating this result, we will first show how to translate the problem of counting fixed points of a mapping into an intersection number problem. Let X be an oriented, compact n-dimensional manifold and $f : X \to X$ a $\mathcal{C}^\infty$ map. Define the *graph of* f in $X \times X$ to be the set

$$\Gamma_f = \{(x, f(x))\,; \quad x \in X\}\,. \tag{5.6.1}$$

It's easy to see that this is an n-dimensional submanifold of $X \times X$ and that this manifold is diffeomorphic to X itself. In fact, in one direction, there is a $\mathcal{C}^\infty$ map

$$\gamma_f : X \to \Gamma_f\,, \quad \gamma_f(x) = (x, f(x))\,, \tag{5.6.2}$$

and, in the other direction, a $\mathcal{C}^\infty$ map

$$\pi : \Gamma_f \to X\,, \quad (x, f(x)) \to x\,, \tag{5.6.3}$$

and it's obvious that these maps are inverses of each other and hence diffeomorphisms. We will orient Γ_f by requiring that γ_f and π be orientation-preserving diffeomorphisms.

An example of a graph is the graph of the identity map of X onto itself. This is the *diagonal* in $X \times X$

$$\Delta = \{(x,x)\,,\ x \in X\} \tag{5.6.4}$$

and its intersection with Γ_f is the set

$$\{(x,x)\,,\ f(x) = x\}\,, \tag{5.6.5}$$

which is just the set of fixed points of f. Hence a natural way to count the fixed points of f is as the intersection number of Γ_f and Δ in $X \times X$. To do so we need these three manifolds to be oriented, but, as we noted above, Γ_f and Δ acquire orientations from the identifications (5.6.2) and, as for $X \times X$, we'll give it its natural orientation as a product of oriented manifolds. (See §4.5.)

Definition 5.6.1. *The global Lefshetz number of X is the intersection number*

$$L(f) = I(\Gamma_f, \Delta)\,. \tag{5.6.6}$$

In this section we'll give two recipes for computing this number: one by topological methods and the other by making transversality assumptions and computing this number as a sum of local intersection numbers a la (5.5.24). We'll first show what one gets from the transversality approach.

Definition 5.6.2. *The map, f, is a Lefshetz map if Γ_f and Δ intersect transversally.*

Let's see what being Lefshetz entails. Suppose p is a fixed point of f. Then at $q = (p,p) \in \Gamma_f$

$$T_q(\Gamma_f) = (d\gamma_f)_p T_p X = \{(\mathrm{v},\, df_p(\mathrm{v}))\,,\ \mathrm{v} \in T_p X\} \tag{5.6.7}$$

and, in particular, for the identity map,

$$T_q(\Delta) = \{(\mathrm{v},\mathrm{v})\,,\ \mathrm{v} \in T_p X\}\,. \tag{5.6.8}$$

Therefore, if Δ and Γ_f are to intersect transversally, the intersection of (5.6.7) and (5.6.8) inside $T_q(X \times X)$ has to be the zero space. In other words if

$$(\mathrm{v}, df_p(\mathrm{v})) = (\mathrm{v},\mathrm{v}) \tag{5.6.9}$$

then $\mathrm{v} = 0$. But the identity (5.6.9) says that v is a fixed point of df_p, so transversality at p amounts to the assertion

$$df_p(\mathrm{v}) = \mathrm{v} \Leftrightarrow \mathrm{v} = 0\,, \tag{5.6.10}$$

or in other words the assertion that the map

$$(I - df_p) : T_pX \to T_pX \tag{5.6.11}$$

is bijective. We'll now prove

Proposition 5.6.3. *The local intersection number $I_p(\Gamma_f, \Delta)$ is* 1 *if (5.6.11) is orientation-preserving and* -1 *if not.*

In other words $I_p(\Gamma_f, \Delta)$ is the sign of $\det(I - df_p)$. To prove this let $e_1, \dots, e_n$ be an oriented basis of T_pX and let

$$df_p(e_i) = \sum a_{j,i} e_j\,. \tag{5.6.12}$$

Now set

$$\mathrm{v}_i = (e_i, 0) \in T_q(X \times X)$$

and

$$w_i = (0, e_i) \in T_q(X \times X)\,.$$

Then by the deifnition of the product orientation on $X \times X$

$$\mathrm{v}_1, \dots, \mathrm{v}_n\,,\ w_1, \dots, w_n \tag{5.6.13}$$

is an oriented basis of $T_q(X \times X)$ and by (5.6.7)

$$\mathrm{v}_1 + \sum a_{j,i} w_j \dots, \mathrm{v}_n + \sum a_{j,n} w_j \tag{5.6.14}$$

is an oriented basis of $T_q\Gamma_f$ and

$$\mathrm{v}_1 + w_1, \dots, \mathrm{v}_n + w_n \tag{5.6.15}$$

is an oriented basis of $T_q\Delta$. Thus $I_p(\Gamma_f, \Delta) = +1$ or -1 depending on whether or not the basis

$$\mathrm{v}_1 + \sum a_{j,i} w_j, \dots, \mathrm{v}_n + \sum a_{j,n} w_j, \mathrm{v}_1 + w_1, \dots, \mathrm{v}_n + w_n$$

of $T_q(X \times X)$ is compatibly oriented with the basis (5.6.12). Thus $I_p(\Gamma_f, \Delta) = +1$ or -1 depending on whether the determinant of the $2n \times 2n$ matrix relating these two bases:

$$\begin{bmatrix} I & A \\ I\,, & I \end{bmatrix}, \quad A = [a_{i.j}] \tag{5.6.16}$$

is positive or negative. However, it's easy to see that this determinant is equal to $\det(I - A)$ and hence by (5.6.12) to $\det(I - df_p)$. *Hint:* By elementary row operations (5.6.16) can be converted into the matrix

$$\begin{bmatrix} I\,, & A \\ 0\,, & I - A \end{bmatrix}.$$

□

Let's summarize what we've shown so far.

Theorem 5.6.4. *The map, $f : X \to X$, is a Lefshetz map if and only if, for every fixed point, p, the map*

$$I - df_p : T_pX \to T_pX \tag{$*$}$$

is bijective. Moreover for Lefshetz maps

$$L(f) = \sum_{p-f_{(p)}} L_p(f) \tag{5.6.17}$$

where $L_p(f) = +1$ if $(*)$ is orientation-preserving and -1 if it's orientation-reversing.

We'll next describe how to compute $L(f)$ as a topological invariant of f. Let ι_Γ be the inclusion map of Γ_f into $X \times X$ and let $T_\Delta \in H^n(X \times X)$ be the Thom class of Δ. Then by (5.5.20)

$$L(f) = I_{\Gamma_f}(\iota^* T_\Delta)$$

and hence since the mapping, $\gamma_f : X \to X \times X$ defined by (5.6.2) is an orientation-preserving diffeomorphism of X onto Γ_f

$$L(f) = I_X(\gamma_f^* T_\Delta)\,. \tag{5.6.18}$$

To evaluate the expression on the right we'll need to know some facts about the cohomology groups of product manifolds. The main result on this topic is the "Künneth" theorem, and we'll take up the formulation and proof of this theorem in §5.7. First, however, we'll describe a result which follows from the Künneth theorem and which will enable us to complete our computation of $L(f)$.

Let π_1 and π_2 be the projection of $X \times X$ onto its first and second factors, i.e., let

$$\pi_i : X \times X \to X \qquad i = 1, 2$$

be the map, $\pi_i(x_1, x_2) = x_i$. Then by (5.6.2)

$$\pi_1 \cdot \gamma_f = i\,d_X \tag{5.6.19}$$

and

$$\pi_2 \cdot \gamma_f = f\,. \tag{5.6.20}$$

Lemma 5.6.5. *If ω_1 and ω_2 are in $\Omega^n(X)$ then*

$$\int_{X \times X} \pi_1^*\omega_1 \wedge \pi_2^*\omega_2 = \left(\int_X \omega_1\right)\left(\int_X \omega_2\right)\,. \tag{5.6.21}$$

Proof. By a partition of unity argument we can assume that ω_i has compact support in a parametrizable open set, V_i. Let U_i be an open subset of $\mathbb{R}^n$ and $\varphi_i : U_i \to V_i$ an orientation-preserving diffeomorphism. Then

$$\varphi_i^*\omega = \rho_i\,dx_1 \wedge \cdots \wedge\, dx_n$$

with $\rho_i \in \mathcal{C}_0^\infty(U_i)$, so the right hand side of (5.6.21) is the product of integrals over $\mathbb{R}^n$:

$$\int \rho_1(x)\,dx \int \rho_2(x)\,dx\,. \tag{5.6.22}$$

Moreover, since $X \times X$ is oriented by its product orientation, the map

$$\psi : U_1 \times U_2 \to V_1 \times V_2$$

mapping (x, y) to $(\varphi_1(x)\,,\,\varphi_2(y))$ is an orientation-preserving diffeomorphism and since $\pi_i \circ \psi = \varphi_i$

$$\begin{aligned}\psi^*(\pi_1^*\omega_1 \wedge \pi_2^*\omega_2) &= \varphi_1^*\omega_1 \wedge \varphi_2^*\omega_2 \\ &= \rho_1(x)\rho_2(y)\,dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n\end{aligned}$$

and hence the left hand side of (5.6.21) is the integral over $\mathbb{R}^{2n}$ of the function, $\rho_1(x)\rho_2(y)$, and therefore, by integration by parts, is equal to the product (5.6.22).

□

As a corollary of this lemma we get a product formula for cohomology classes:

Lemma 5.6.6. *If c_1 and c_2 are in $H^n(X)$ then*

$$I_{X\times X}(\pi_1^* c_1 \cdot \pi_2^* c_2) = I_X(c_1) I_X(c_2)\,. \tag{5.6.23}$$

Now let $d_k = \dim H^k(X)$ and note that since X is compact, Poincaré duality tells us that $d_k = d_\ell$ when $\ell = n-k$. In fact it tells us even more. Let

$$\mu_i^k\,, \quad i = 1, \ldots, d_k$$

be a basis of $H^k(X)$. Then, since the pairing (5.4.9) is non-singular, there exists for $\ell = n-k$ a "dual" basis

$$\nu_j^\ell\,, \quad j = 1, \ldots, d_\ell$$

of $H^\ell(X)$ satisfying

$$I_X(\mu_i^k \cdot \nu_j^\ell) = \delta_{ij}\,. \tag{5.6.24}$$

Lemma 5.6.7. *The cohomology classes*

$$\pi_1^\sharp \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k\,, \quad k+\ell = n \tag{5.6.25}$$

for $k = 0, \ldots, n$ and $1 \le r, s \le d_k$, are a basis for $H^n(X \times X)$.

This is the corollary of the Künneth theorem that we alluded to above (and whose proof we'll give in §5.7). Using these results we'll prove

Theorem 5.6.8. *The Thom class, T_Δ, in $H^n(X \times X)$ is given explicitly by the formula*

$$T_\Delta = \sum_{k+\ell=n} (-1)^\ell \sum_{i=1}^{d_k} \pi_1^\sharp \mu_i^k \cdot \pi_2^\sharp \nu_i^\rho\,. \tag{5.6.26}$$

Proof. We have to check that for every cohomology class, $c \in H^n(X \times X)$, the class, T_Δ, defined by (5.6.26) has the reproducing property

$$I_{X\times X}(T_\Delta \cdot c) = I_\Delta(\iota_\Delta^\sharp c) \tag{5.6.27}$$

where ι_Δ is the inclusion map of Δ into $X \times X$. However the map

$$\gamma_\Delta : X \to X \times X\,, \quad x \to (x, x)$$

is an orientation-preserving diffeomorphism of X onto Δ, so it suffices to show that

$$I_{X\times X}(T_\Delta \cdot c) = I_X(\gamma_\Delta^\sharp c) \tag{5.6.28}$$

and by Lemma 5.6.7 it suffices to verify (5.6.28) for c's of the form

$$c = \pi_1^\sharp \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k\,.$$

The product of this class with a typical summand of (5.6.26), for instance, the summand

$$(-1)^{\ell'} \pi_1^\sharp \mu_i^{k'} \cdot \pi_2^\sharp \nu_i^{\ell'}\,, \quad k' + \ell' = n\,, \tag{5.6.29}$$

is equal, up to sign to,

$$\pi_1^\sharp \mu_i^{k'} \cdot \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k \cdot \nu_i^{\ell'}\,.$$

Notice, however, that if $k \neq k'$ this product is zero: For $k < k'$, $k' + \ell$ is greater than $k + \ell$ and hence greater than n. Therefore

$$\mu_i^{k'} \cdot \nu_r^\ell \in H^{k'+\ell}(X)$$

is zero since X is of dimension n, and for $k > k'$, ℓ' is greater than ℓ and $\mu_s^k \cdot \nu_i^{\ell'}$ is zero for the same reason. Thus in taking the product of T_Δ with c we can ignore all terms in the sum except for the terms, $k' = k$ and $\ell' = \ell$. For these terms, the product of (5.6.29) with c is

$$(-1)^{k\ell} \pi_1^\sharp \mu_i^k \cdot \nu_r^\ell \cdot \pi_2^\sharp \mu_3^k \cdot \nu_i^\ell\,.$$

(Exercise: Check this. *Hint:* $(-1)^\ell (-1)^{\ell^2} = 1$.) Thus

$$T_\Delta \cdot c = (-1)^{k\ell} \sum_i \pi_1^\sharp \mu_i^k \cdot \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k \cdot \nu_i^\ell$$

and hence by Lemma 5.6.5 and (5.6.24)

$$\begin{aligned} I_{X\times X}(T_\Delta \cdot c) &= (-1)^{k\ell} \sum_i I_X(\mu_i^k \cdot \nu_r^\ell) I_X(\mu_s^k \cdot \nu_i^\ell) \\ &= (-1)^{k\ell} \sum_i \delta_{ir}\delta_{is} \\ &= (-1)^{k\ell} \delta_{rs}\,. \end{aligned}$$

On the other hand for $c = \pi_1^\sharp \nu_r^\ell \cdot \pi_2^\sharp \mu_s^k$

$$\begin{aligned}\gamma_\Delta^\sharp c &= \gamma_\Delta^\sharp \pi_1^\sharp \nu_r^\ell \cdot \gamma_\Delta^\sharp \pi_2^\sharp \mu_s^k \\ &= (\pi_1 \cdot \gamma_\Delta)^\sharp \nu_r^\ell (\pi_2 \cdot \gamma_\Delta)^\sharp \mu_s^k \\ &= \nu_r^\ell \cdot \mu_s^k\end{aligned}$$

since

$$\pi_1 \cdot \nu_\Delta = \pi_2 \cdot \gamma_\Delta = id_X \,.$$

So

$$I_X(\gamma_\Delta^\sharp c) = I_X(\nu_r^\ell \cdot \mu_s^k) = (-1)^{k\ell}\delta_{rs}$$

by (5.6.24). Thus the two sides of (5.6.27) are equal.

□

We're now in position to compute $L(f)$, i.e., to compute the expression $I_X(\gamma_f^* T_\Delta)$ on the right hand side of (5.6.18). Since ν_i^ℓ, $i = 1, \ldots, d_\ell$ is a basis of $H^\ell(X)$ the linear mapping

(5.6.30) $$f^\sharp : H^\ell(X) \to H^\ell(X)$$

can be described in terms of this basis by a matrix, $[f_{ij}^\ell]$ with the defining property

$$f^\sharp \nu_i^\ell = \sum f_{ji}^\ell \nu_j^\ell \,.$$

Thus by (5.6.26), (5.6.19) and (5.6.20)

$$\begin{aligned}\gamma_f^\sharp T_\Delta &= \gamma_f^\sharp (-1)^\ell \sum_{k+\ell=n} \sum_i \pi_1^\sharp u_i^k \cdot \pi_2^\sharp \nu_i^\ell \\ &= \sum (-1)^\ell \sum_i (\pi_1 \cdot \gamma_f)^\sharp \mu_i^k \cdot (\pi_2 \cdot \nu_f)^\sharp \nu_i^\ell \\ &= \sum (-1)^\ell \sum \mu_i^k \cdot f^\sharp \nu_i^\ell \\ &= \sum (-1)^\ell \sum f_{ji}^\ell \mu_i^k \cdot \nu_j^\ell \,.\end{aligned}$$

Thus by (5.6.24)

$$\begin{aligned}I_X(\gamma_f^\sharp T_\Delta) &= \sum (-1)^\ell \sum f_{ji}^\ell I_X(\mu_i^k \cdot \nu_j^\ell) \\ &= \sum (-1)^\ell \sum f_{ji}^\ell \delta_{ij} \\ &= \sum_{\ell=0}^n (-1)^\ell \left(\sum_i f_{ii}^\ell \right) \,.\end{aligned}$$

But $\sum_i f^\ell_{i,i}$ is just the trace of the linear mapping (5.6.30) (see exercise 12 below), so we end up with the following purely topological prescription of $L(f)$.

Theorem 5.6.9. *The Lefshetz number, $L(f)$ is the alternating sum*

$$\sum (-1)^\ell \operatorname{Trace}(f^\sharp)_\ell \tag{5.6.31}$$

where $\operatorname{Trace}(f^\sharp)_\ell$ *is the trace of the mapping*

$$f^\sharp : H^\ell(X) \to H^\ell(X) \,.$$

Exercises.

1. Show that if $f_0 : X \to X$ and $f_1 : X \to X$ are homotopic $\mathcal{C}^\infty$ mappings $L(f_0) = L(f_1)$.

2. (a) The Euler characteristic, $\chi(X)$, of X is defined to be the intersection number of the diagonal with itself in $X \times X$, i.e., the "self-intersection" number

$$I(\Delta, \Delta) = I_{X\times X}(T_\Delta, T_\Delta) \,.$$

Show that if a $\mathcal{C}^\infty$ map, $f : X \to X$ is homotopic to the identity, $L_f = \chi(X)$.

(b) Show that

$$\chi(X) = \sum_{\ell=0}^{n} (-1)^\ell \dim H^\ell(X) \,. \tag{5.6.32}$$

(c) Show that $\chi(X) = 0$ if n is odd.

3. (a) Let S^n be the unit n-sphere in $\mathbb{R}^{n+1}$. Show that if $g : S^n \to S^n$ is a $\mathcal{C}^\infty$ map

$$L(g) = 1 + (-1)^n (\deg)\,(\mathrm{g}) \,.$$

(b) Conclude that if $\deg(g) \neq (-1)^{n+1}$, then g has to have a fixed point.

4. Let f be a $\mathcal{C}^\infty$ mapping of the closed unit ball, B^{n+1}, into itself and let $g : S^n \to S^n$ be the restriction of f to the boundary of B^{n+1}. Show that if $\deg(g) \neq (-1)^{n+1}$ then the fixed point of f predicted by Brouwer's theorem can be taken to be a point on the boundary of B^{n+1}.

5. (a) Show that if $g : S^n \to S^n$ is the antipodal map, $g(x) = -x$, then $\deg(g) = (-1)^{n+1}$.

(b) Conclude that the result in #4 is sharp. Show that the map

$$f : B^{n+1} \to B^{n+1}, \quad f(x) = -x,$$

has only one fixed point, namely the origin, and in particular has no fixed points on the boundary.

6. Let v be a vector field on X. Since X is compact, v generates a one-parameter group of diffeomorphisms

$$f_t : X \to X, \quad -\infty < t < \infty. \tag{5.6.33}$$

(a) Let $\sum_t$ be the set of fixed points of f_t. Show that this set contains the set of zeroes of v, i.e., the points, $p \in X$ where $v(p) = 0$.

(b) Suppose that for some t_0, f_{t_0} is Lefshetz. Show that for all t, f_t maps $\sum_{t_0}$ into itself.

(c) Show that for $|t| < \epsilon$, ϵ small, the points of $\sum_{t_0}$ are fixed points of f_t.

(d) Conclude that $\sum_{t_0}$ is equal to the set of zeroes of v.

(e) In particular, conclude that for all t the points of $\sum_{t_0}$ are fixed points of f_t.

7. (a) Let V be a finite dimensional vector space and

$$F(t) : V \to V, \; -\infty < t < \infty$$

a one-parameter group of linear maps of V onto itself. Let $A = \frac{dF}{dt}(0)$. Show that $F(t) = \exp tA$. (See §2.1, exercise 7.)

(b) Show that if $I - F(t_0) : V \to V$ is bijective for some t_0, then $A : V \to V$ is bijective. *Hint:* Show that if $A\mathrm{v} = 0$ for some $\mathrm{v} \in V - \{0\}$, $F(t)\mathrm{v} = \mathrm{v}$.

8. Let v be a vector field on X and let (5.6.33) be the one-parameter group of diffeomorphisms generated by v. If $v(p) = 0$ then by part (a) of exercise 6, p is a fixed point of f_t for all t.

(a) Show that

$$(df_t) : T_pX \to T_pX$$

is a one-parameter group of linear mappings of T_pX onto itself.

(b) Conclude from #7 that there exists a linear map

$$L_v(p) : T_pX \to T_pX \tag{5.6.34}$$

with the property

$$\exp tL_v(p) = (df_t)_p\,. \tag{5.6.35}$$

9. Suppose f_{t_0} is a Lefshetz map for some t_0. Let $a = t_0/N$ where N is a positive integer. Show that f_a is a Lefshetz map. *Hints:*

(a) Show that

$$f_{t_0} = f_a \circ \cdots \circ f_a = f_a^N$$

(i.e., f_a composed with itself N times).

(b) Show that if p is a fixed point of f_a, it is a fixed point of f_{t_0}.

(c) Conclude from exercise 6 that the fixed points of f_a are the zeroes of v.

(d) Show that if p is a fixed point of f_a,

$$(df_{t_0})_p = (df_a)_p^N\,.$$

(e) Conclude that if $(df_a)_p\mathrm{v} = \mathrm{v}$ for some $\mathrm{v} \in T_pX - \{0\}$, then $(df_{t_0})_p\mathrm{v} = \mathrm{v}$.

10. Show that for all t, $L(f_t) = \chi(X)$. *Hint:* Exercise 2.

11. (The Hopf theorem.) A vector field v on X is a *Lefshetz* vector field if for some t_0, f_{t_0} is a Lefshetz map.

(a) Show that if v is a Lefshetz vector field then it has a finite number of zeroes and for each zero, p, the linear map (5.6.34) is bijective.

(b) For a zero, p, of v let $\sigma_p(v) = +1$ if the map (5.6.34) is orientation-preserving and -1 if it's orientation-reversing. Show that

$$\chi(X) = \sum_{v(p)=0} \sigma_p(v)\,.$$

Hint: Apply the Lefshetz theorem to f_a, $a = t_0/N$, N large.

12. (The trace of a linear mapping: a quick review.)

For $A = [a_{i,j}]$ an $n \times n$ matrix define

$$\text{trace}\, A = \sum a_{i,i}\,.$$

(a) Show that if A and B are $n \times n$ matrices

$$\text{trace}\, AB = \text{trace}\, BA\,.$$

(b) Show that if B is an invertible $n \times n$ matrix

$$\text{trace}\, BAB^{-1} = \text{trace}\, A\,.$$

(c) Let V be and n-dimensional vector space and $L : V \to V$ a liner map. Fix a basis $\mathrm{v}_1, \ldots, \mathrm{v}_n$ of V and define the trace of L to be the trace of A where A is the defining matrix for L in this basis, i.e.,

$$L\mathrm{v}_i = \sum a_{j,i}\mathrm{v}_j\,.$$

Show that this is an *intrinsic* definition not depending on the basis $\mathrm{v}_1, \ldots, \mathrm{v}_n$.

5.7 The Künneth theorem

Let X be an n-dimensional manifold and Y an r-dimensional manifold, both of these manifolds having finite topology. Let

$$\pi : X \times Y \to X$$

be the projection map, $\pi(x, y) = x$ and

$$\rho : X \times Y \to Y$$

the projection map $(x, y) \to y$. Since X and Y have finite topology their cohomology groups are finite dimensional vector spaces. For $0 \le k \le n$ let

$$\mu_i^k\,, \quad 1 \le i \le \dim H^k(X)\,,$$

be a basis of $H^k(X)$ and for $0 \le \ell \le r$ let

$$\nu_j^\ell\,, \quad 1 \le j \le \dim H^\ell(Y)$$

be a basis of $H^\ell(Y)$. Then for $k + \ell = m$ the product, $\pi^\sharp \mu_i^k \cdot \rho^\sharp \nu_j^\ell$, is in $H^m(X \times Y)$. The Künneth theorem asserts

Theorem 5.7.1. *The product manifold, $X \times Y$, has finite topology and hence the cohomology groups, $H^m(X \times Y)$ are finite dimensional. Moreover, the products over $k + \ell = m$*

$$\pi^\sharp \mu_i^k \cdot \rho^\sharp \nu_j^\ell\,,\ 0 \le i \le \dim H^k(X)\,,\ 0 \le j \le \dim H^\ell(Y)\,, \tag{5.7.1}$$

are a basis for the vector space $H^m(X \times Y)$.

The fact that $X \times Y$ has finite topology is easy to verify. If U_i, $i = 1, \ldots, M$, is a good cover of X and V_j, $j = 1, \ldots, N$, is a good cover of Y the products of these open sets, $U_i \times U_j$, $1 \le i \le M$, $1 \le j \le N$ is a good cover of $X \times Y$: For every multi-index, I, U_I is either empty or diffeomorphic to $\mathbb{R}^n$, and for every multi-index, J, V_J is either empty or diffeomorphic to $\mathbb{R}^r$, hence for any product multi-index (I, J), $U_I \times V_J$ is either empty or diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^r$. The tricky part of the proof is verifying that the products, (5.7.1) are a basis of $H^m(X \times Y)$, and to do this it will be helpful to state the theorem above in a form that avoids our choosing specified bases for $H^k(X)$ and $H^\ell(Y)$. To do so we'll need to generalize slightly the notion of a bilinear pairing between two vector space.

Definition 5.7.2. *Let V_1, V_2 and W be finite dimensional vector spaces. A map $B : V_1 \times V_2 \to W$ is a bilinear map if it is linear in each of its factors, i.e., for $\mathrm{v}_2 \in V_2$ the map*

$$\mathrm{v} \in V_1 \to B(\mathrm{v}_1, \mathrm{v}_2)$$

is a linear map of V_1 into W and for $\mathrm{v}_1 \in V_1$ so is the map

$$\mathrm{v} \in V_2 \to B(\mathrm{v}_1, \mathrm{v})\,.$$

It's clear that if B_1 and B_2 are bilinear maps of $V_1 \times V_2$ into W and λ_1 and λ_2 are real numbers the function

$$\lambda_1 B_1 + \lambda_2 B_2 : V_1 \times V_2 \to W$$

is also a bilinear map of $V_1 \times V_2$ into W, so the set of all bilinear maps of $V_1 \times V_2$ into W forms a vector space. In particular the set of all bilinear maps of $V_1 \times V_2$ into $\mathbb{R}$ is a vector space, and since this vector space will play an essential role in our intrinsic formulation of the Künneth theorem, we'll give it a name. We'll call it the *tensor product* of V_1^* and V_2^* and denote it by $V_1^* \otimes V_2^*$. To explain where this terminology comes from we note that if ℓ_1 and ℓ_2 are vectors in V_1^* and V_2^* then one can define a bilinear map

$$\ell_1 \otimes \ell_2 : V_1 \times V_2 \to \mathbb{R} \tag{5.7.2}$$

by setting $(\ell_1 \otimes \ell_2)(\mathrm{v}_1, \mathrm{v}_2) = \ell_1(\mathrm{v}_1)\ell_2(\mathrm{v}_2)$. In other words one has a tensor product map:

$$V_1^* \times V_2^* \to V_1^* \otimes V_2^* \tag{5.7.3}$$

mapping (ℓ_1, ℓ_2) to $\ell_1 \otimes \ell_2$. We leave for you to check that this is a bilinear map of $V_1^* \times V_2^*$ into $V_1^* \otimes V_2^*$ and to check as well

Proposition 5.7.3. *If ℓ_i^1, $i = 1, \dots, m$ is a basis of V_1^* and ℓ_j^2, $j = 1, \dots, n$ is a basis of V_2^* then $\ell_i^1 \otimes \ell_j^2$, $1 \le i \le m$, $1 \le j \le n$, is a basis of $V_1^* \otimes V_2^*$.*

Hint: If V_1 and V_2 are the same vector space you can find a proof of this in §1.3 and the proof is basically the same if they're different vector spaces.

Corollary 5.7.4. *The dimension of $V_1^* \otimes V_2^*$ is equal to the dimension of V_1^* times the dimension of V_2^*.*

We'll now perform some slightly devious maneuvers with "duality" operations. First note that for any finite dimensional vector space, V, the pairing

$$V \times V^* \to \mathbb{R}, \quad (\mathrm{v}, \ell) \to \ell(\mathrm{v}) \tag{5.7.4}$$

is a non-singular bilinear pairing, so, as we explained in §5.4 it gives rise to a bijective linear mapping

$$V \to (V^*)^*. \tag{5.7.5}$$

Next note that if

$$L : V_1 \times V_2 \to W \tag{5.7.6}$$

is a bilinear mapping and $\ell : W \to \mathbb{R}$ a linear mapping (i.e., an element of W^*), then the composition of ℓ and L is a bilinear mapping

$$\ell \circ L : V_1 \times V_2 \to \mathbb{R}$$

and hence by definition an element of $V_1^* \otimes V_2^*$. Thus from the bilinear mapping (5.7.6) we get a *linear* mapping

$$L^\sharp : W^* \to V_1^* \otimes V_2^* \,. \tag{5.7.7}$$

We'll now define a notion of tensor product for the vector spaces V_1 and V_2 themselves.

Definition 5.7.5. *The vector space, $V_1 \otimes V_2$ is the vector space dual of $V_1^* \otimes V_2^*$, i.e., is the space*

$$V_1 \otimes V_2 = (V_1^* \otimes V_2^*)^* \,. \tag{5.7.8}$$

One implication of (5.7.8) is that there is a natural bilinear map

$$V_1 \times V_2 \to V_1 \otimes V_2 \,. \tag{5.7.9}$$

(In (5.7.3) replace V_i by V_i^* and note that by (5.7.5) $(V_i^*)^* = V_i$.) Another is the following:

Proposition 5.7.6. *Let L be a bilinear map of $V_1 \times V_2$ into W. Then there exists a unique linear map*

$$L^\# : V_1 \otimes V_2 \to W \tag{5.7.10}$$

with the property

$$L^\#(\mathrm{v}_1 \otimes \mathrm{v}_2) = L(\mathrm{v}_1, \mathrm{v}_2) \tag{5.7.11}$$

where $\mathrm{v}_1 \otimes \mathrm{v}_2$ is the image of $(\mathrm{v}_1, \mathrm{v}_2)$ with respect to (5.7.9).

Proof. Let $L^\#$ be the transpose of the map $L^\sharp$ in (5.7.7) and note that by (5.7.5) $(W^*)^* = W$.

□

Notice that by Proposition 5.7.6 the property (5.7.11) is the *defining* property of $L^{\#}$, it uniquely determines this map. (This is in fact the whole point of the tensor product construction. Its purpose is to convert bilinear objects into linear objects.)

After this brief digression (into an area of mathematics which some mathematicians unkindly refer to as "abstract nonsense") let's come back to our motive for this digression: an intrinsic formulation of the Künneth theorem. As above let X and Y be manifolds of dimension n and r, respectively, both having finite topology. For $k+\ell=m$ one has a bilinear map

$$H^k(X)\times H^\ell(Y)\to H^m(X\times Y)$$

mapping (c_1,c_2) to $\pi^*c_1\cdot\rho^*c_2$, and hence by Proposition 5.7.6 a *linear* map

$$H^k(X)\otimes H^\ell(Y)\to H^m(X\times Y)\,. \tag{5.7.12}$$

Let

$$H_1^m(X\times Y)=\sum_{k+\ell=m}H^k(X)\otimes H^\ell(Y)\,.$$

The maps (5.7.12) can be combined into a single linear map

$$H_1^m(X\times Y)\to H^m(X\times Y) \tag{5.7.13}$$

and our intrinsic version of the Künneth theorem asserts

Theorem 5.7.7. *The map (5.7.13) is bijective.*

Here is a sketch of how to prove this. (Filling in the details will be left as a series of exercises.) Let U be an open subset of X which has finite topology and let

$$\mathcal{H}_1^m(U)=\sum_{k+\ell=m}H^k(U)\otimes H^\ell(Y)$$

and

$$\mathcal{H}_2^m(U)=H^m(U\times Y)\,.$$

As we've just seen there's a Künneth map

$$\kappa:\mathcal{H}_1^m(U)\to\mathcal{H}_2^m(U)\,.$$

Exercises.

1. Let U_1 and U_2 be open subsets of X, both having finite topology, and let $U = U_1 \cup U_2$. Show that there is a long exact sequence:

$$\xrightarrow{\delta} \mathcal{H}_1^m(U) \longrightarrow \mathcal{H}_1^m(U_1) \oplus \mathcal{H}_1^m(U_2) \longrightarrow \mathcal{H}_1^m(U_1 \cap U_2) \xrightarrow{\delta} \mathcal{H}_1^{m+1}(U) \longrightarrow$$

Hint: Take the usual Mayer–Victoris sequence:

$$\xrightarrow{\delta} H^k(U) \longrightarrow H^k(U_1) \oplus H^k(U_2) \longrightarrow H^k(U_1 \cap U_2) \xrightarrow{\delta} H^{k+1}(U) \longrightarrow$$

tensor each term in this sequence with $H^\ell(Y)$ and sum over $k + \ell = m$.

2. Show that for $\mathcal{H}_2$ there is a similar sequence. *Hint:* Apply Mayer–Victoris to the open subsets $U_1 \times Y$ and $U_2 \times Y$ of M.

3. Show that the diagram below commutes. (This looks hard but is actually very easy: just write down the definition of each arrow in the language of forms.)

$$\begin{array}{ccccccccc}
\xrightarrow{\delta} & \mathcal{H}_2^m(U) & \longrightarrow & \mathcal{H}_2^m(U_1) \oplus \mathcal{H}_2^m(U_2) & \longrightarrow & \mathcal{H}_2^m(U_1 \cap U_2) & \xrightarrow{\delta} & \mathcal{H}_2^{m+1}(U) & \longrightarrow \\
 & k\uparrow & & k\uparrow & & k\uparrow & & k\uparrow & \\
\xrightarrow{\delta} & \mathcal{H}_1^m(U) & \longrightarrow & \mathcal{H}_1^m(U_1) \oplus \mathcal{H}_1^m(U_2) & \longrightarrow & \mathcal{H}_1^m(U_1 \cap U_2) & \xrightarrow{\delta} & \mathcal{H}_1^{m+1}(U) & \longrightarrow
\end{array}$$

4. Conclude from Exercise 3 that if the Künneth map is bijective for U_1, U_2 and $U_1 \cap U_2$ it is bijective for U.

5. Prove the Künneth theorem by induction on the number of open sets in a good cover of X. To get the induction started, note that

$$H^k(X \times Y) \cong H^k(Y)$$

if $X = \mathbb{R}^n$. (See §5.3, exercise 11.)